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An Algorithm for Constructing a Planar Layout of a Graph with a Regular Polygon as Outer Face

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Abstract

Read's algorithm for constructing a planar layout of a graph G produces a straight-line embedding of G, by using a sequence of triangulations. Let F denote any face of G. In this paper, Read's algorithm is modified. A straight-line embedding is constructed in which F forms the outer face, such that its vertices lie on a convex regular polygon. It is proved that the method always works. Usually F is taken as the face of largest degree. The complexity of the algorithm is linear in the number of vertices of G.

1. Read's Algorithm

Let G be a planar, 2-connected, undirected, simple graph on $n \geq 4$ vertices. The vertex and edge sets of G are $V(G)$ and $E(G)$. If $u, v \in V(G)$, then $u \rightarrow v$ means that u is adjacent to v (and so also $v \rightarrow u$). The reader is referred to Bondy and Murty [1] for other graph-theoretic terminology. We begin with a brief description of Read's algorithm for finding a planar layout of a graph. See Read [5] for more detailed information. G is known to be planar, and we assume that initially we are given the clockwise cyclic ordering of the edges at each vertex in a planar embedding. This is sufficient to define the faces of the embedding. The algorithm of [4] will construct such a clockwise ordering of the edges at each vertex. If G is not a triangulation, then we can complete it to a triangulation on n vertices by adding appropriate diagonal edges in some of the faces of G. This gives a triangulation which we shall call G_n. Read's algorithm then proceeds to reduce G_n to a triangulation G_{n-1} on $n-1$ vertices by deleting some vertex. The reduction works as follows. Let u be a vertex of G. If $\deg(u) = 3$, then $G_{n-1} = G_n - u$ is a triangulation on $n-1$ vertices. See Fig. 1. If $\deg(u) = 4$, let v, w, x, y be

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the vertices adjacent to u, in that order. Then (v, w, x, y) is a quadrilateral face in $G_n - u$. If $v \not\to x$ then $G_{n-1} = G_n - u + vx$ is a triangulation on $n-1$ vertices. But if $v \to x$ then $w \not\to y$, and $G_{n-1} = G_n - u + wy$ is a triangulation on $n-1$ vertices. See Fig. 2.

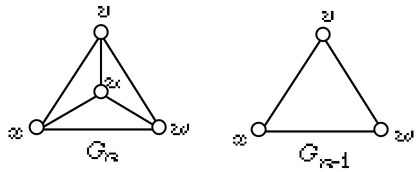

Fig. 1, $\deg(u) = 3$

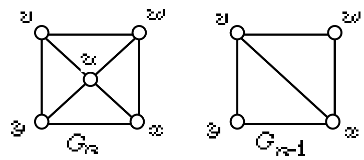

Fig. 2, $\deg(u) = 4$

Finally, if $\deg(u) = 5$, let v, w, x, y, z be the vertices adjacent to u, which form a pentagonal face in $G_n - u$. We can triangulate this face by adding the two diagonals vx and vy, unless one of these is already an edge of G_n. See Fig. 3. If one of these, say vx is an existing edge, then wy and wz are two diagonals that are not edges of G_n. Similarly if vy is an existing edge of G_n. It follows that the pentagonal face can always be triangulated, giving a triangulation G_{n-1}.

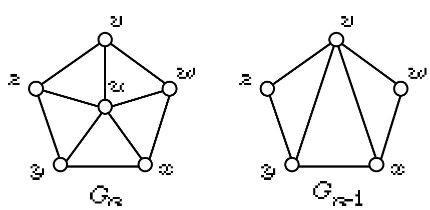

Fig. 3, $\deg(u) = 5$

The following simple observation is important to the success of the algorithm.

1.1. Let a vertex u be deleted from a triangulation G_n, as above, to produce a triangulation G_{n-1} by adding up to two diagonals in the non-triangular face of $G_n - u$. If $\deg(u) = 3$, then all vertices adjacent to u decrease their degree by one. If $\deg(u) = 4$, then two vertices adjacent to u decrease their degree by one, the other two remain unchanged. If $\deg(u) = 5$, then two vertices adjacent to u decrease their degree by one, two remain unchanged, and one increases by one.

Let G_n have n vertices, ε edges, and f faces. Since G_n is a triangulation, we have $3f = 2\varepsilon$. Then from Euler's formula,

$$n - \varepsilon + f = 2,$$

it follows that $3n - \varepsilon = 6$. Let n_3, n_4, n_5, ... denote the number of vertices of G_n of degree 3, 4, 5, etc. Since $n \geq 4$ and G is simple, there are no vertices of degree 2 or less. Then $n = n_3 + n_4 + n_5 + \ldots$ and $2\varepsilon = 3n_3 + 4n_4 + 5n_5 + \ldots$. Substituting these into the formula above gives $3n_3 + 2n_4 + n_5 = 12 + n_7 + 2n_8 + 3n_9 + \ldots$. We will refer to this often. Therefore we note it as follows.

1.2. Let G be a simple triangulation on $n \geq 4$ vertices with ε edges. Let n_k denote the number of vertices of degree k. Then $6n - 2\varepsilon = 12$ and $3n_3 + 2n_4 + n_5 = 12 + n_7 + 2n_8 + 3n_9 + \ldots$.

It follows that G_n always contains a vertex of degree 3, 4, or 5. Therefore the reduction from G_n to G_{n-1} can always be accomplished. Read's algorithm then continues this reduction until G_3 is reached. Since G_3 is a triangle, its vertices can be placed anywhere in the plane, so some placement is chosen. The reduction process is then reversed, and the deleted nodes are reinserted in the reverse order to which they were deleted. Each node is placed inside a triangle, quadrilateral, or pentagon. Read shows how to do this [5] in such a way that a straight-line embedding of G_n is constructed. The edges added to G in order to triangulate it are then deleted, leaving an embedding of the original graph G.

We have programmed this algorithm, and found that it produces embeddings which tend to squash the majority of nodes together into a small corner of the graph. The outer face of G_n is always a triangle. When the triangulating edges of G_n are removed in order to get G, this gives the outer face of G an unusual shape. Two typical examples follow in Figs. 4 and 5. Fig. 4 shows the graph of the cube. Fig. 5 is a planar embedding of Tutte's graph. (Tutte's graph was the first known counterexample to Tait's conjecture, that is, it is a planar, 3-connected, trivalent graph that is non-hamiltonian. See Bondy and Murty [1] for further information.)

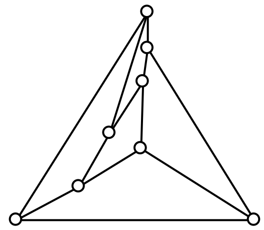

Fig. 4, The graph of the cube

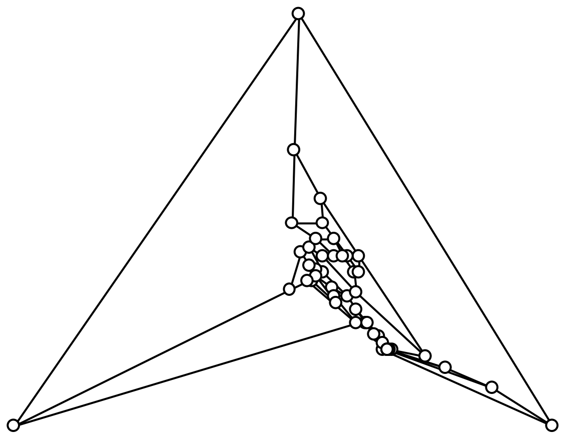

Fig. 5, Tutte's graph

The purpose of this work is to show how to modify Read's algorithm so as to produce planar embeddings in which the outer face is a regular polygon, and in which the nodes do not tend to accumulate into a small area of the embedding.

2. Weighted Averages.

As above, we have G_n is a triangulation of G. By deleting one vertex of G_n at a time, we produce a sequence G_n, G_{n-1}, G_{n-2}, $\ldots$, G_3 of triangulations. From each G_k, $(n \geq k \geq 4)$, a vertex u_k is deleted, and up to two diagonal edges are inserted in the non-triangular face of $G_k - u_k$. This produces G_{k-1}. The accumulation of nodes of G into a small area of the graph occurs because each deleted node u_k is placed in the *centre* of a triangle or quadrilateral when it is restored. (Pentagons are more complicated.) The remaining vertices u_{k+1}, $\ldots$, u_n are thereby often forced into a small number of the faces of G_k. A more equitable positioning of the vertices can be accomplished by taking a suitable weighted average of the coordinates of the vertices of the face in which u_k is placed.

Notice that the assignment of positions to the vertices of G_n produces simultaneous embeddings of all G_k. To each face F of G_k there is a unique clockwise traversal of the boundary of F. This is called the *facial cycle* of F. For each edge $vw \in E(G_k)$, there is a unique face F_{vw} whose facial cycle contains v followed by w. There is a natural coorespondence relating each face of G_n to a unique face of G_k, as we now show. For each $vw \in E(G_k)$, let $f_k(vw)$ denote the number of faces of G_n which correspond to F_{vw} of G_k. It is easy to compute the numbers $f_k(vw)$ when performing the reduction G_n, G_{n-1}, G_{n-2}, etc. This can be done as follows. Initially $f_n(vw) = 1$, for all $vw \in E(G_n)$. Let u be the vertex deleted, and refer to Figs. 1, 2, and 3.

If $\deg(u) = 3$, the three faces uvw, uwx, and uxv of G_k are all contained within vwx in G_{k-1}. Therefore the faces of G_n corresponding to uvw, uwx, and uxv in G_k all correspond to vwx in G_{k-1}. So $f_{k-1}(vw) = f_{k-1}(wx) = f_{k-1}(xv) = f_k(vw) + f_k(wx) + f_k(xv)$.

If $\deg(u) = 4$, the faces uvw and uwx of G_k are contained within vwx of G_{k-1}. The faces uxy and uyv are contained within vxy of G_{k-1}. Therefore the faces of G_n corresponding to uvw and uxw in G_k all correspond to vwx in G_{k-1}. So $f_{k-1}(vw) = f_{k-1}(wx) = f_{k-1}(xv) = f_k(vw) + f_k(wx)$. Similarly $f_{k-1}(vx) = f_{k-1}(xy) = f_{k-1}(yv) = f_k(xy) + f_k(yv)$.

If $\deg(u) = 5$, the faces of G_k within the pentagon $vwxyz$ are not contained within the resulting faces of of G_{k-1}. We arbitrarily choose the faces uvw and uwx of G_k to correspond to vwx of G_{k-1}. Similarly, faces uyz and uzv of G_k correspond to vyz of G_{k-1}. Face uxy of G_k corresponds to vxy of G_{k-1}. Therefore $f_{k-1}(vw) = f_{k-1}(wx) = f_{k-1}(xv) = f_k(vw) + f_k(wx)$, $f_{k-1}(vx) = f_{k-1}(xy) = f_{k-1}(yv) = f_k(xy)$, and $f_{k-1}(vy) = f_{k-1}(yz) = f_{k-1}(zv) = f_k(yz) + f_k(zv)$.

Let $P(a)$ denote the cartesian coordinates in the plane which will be assigned to vertex a, for all $a \in V(G)$. When G_3 is reached, let its vertices be v, w, and x. One of the faces, vxw, will satisfy $f_3(vx) = f_3(xw) = f_3(wv) = 1$. The other face will have $f_3(vw) = f_3(wx) = f_3(xv) = f - 1$, where f is the number of faces of G_n. The vertices v, w and x can be equally spaced along the circumference of a circle, creating an equilateral triangle. The size of it will depend on the area available for drawing G. The remaining vertices are then reinserted in reverse order, G_3, G_4, G_5, …, as follows. We use the labelling of Figs. 1, 2, and 3, where a vertex u is to be restored to G_k to produce G_{k+1}.

2.1. If $\deg(u) = 3$, then let $N = f_k(vw) + f_k(wx) + f_k(xv)$. Define

$$P(u) = \frac{1}{N}\{P(v)f_k(wx) + P(w)f_k(xv) + P(x)f_k(vw)\}$$

Notice that this is a convex combination of $P(v)$, $P(w)$, and $P(x)$. Therefore u will be placed inside the triangle of v, w, and x. It is proved below that this weighted average assigns areas to uvw, uwx, and uxv proportional to the number of faces of G_n corresponding to them.

2.2. If $\deg(u) = 4$, then let $N_v = f_k(vw) + f_k(yv)$ and $N_x = f_k(wx) + f_k(xy)$. Let $N = N_v + N_x$. Define

$$P(u) = \frac{1}{N}\{P(v)N_x + P(x)N_v\}$$

This is a convex combination of $P(v)$ and $P(x)$ which places u on the diagonal edge connecting v to x.

2.3. If $\deg(u) = 5$, then the placement of u in Read's algorithm requires an examination of 9 different cases that can arise concerning the shape of the pentagon $F = vwxyz$. F is not always a convex polygon. We use the same method for placing u as in Read's algorithm. The weights $f_k(vw)$, etc. are not used. See Read [5] for the details.

Using these weighted averages to place the deleted vertex gives the resulting triangles an area proportional to the number of faces of G_n corresponding to the triangle. This is a consequence of the following lemma.

2.4 Lemma. *Let P, Q, and R be three points in the plane, and let S be any point inside the triangle PQR, that is, $S = aP + bQ + cR$, where $a, b, c > 0$ and $a + b + c = 1$. Then $A(SQR) = a.A(PQR)$, $A(SRP) = b.A(PQR)$, and $A(SPQ) = c.A(PQR)$, where $A(PQR)$ denotes the area of triangle PQR, etc.*

Proof. Let (p_1, p_2) denote the coordinates of P, and similarly for Q and R. Then the area of the triangle PQR is one half the determinant of the matrix whose rows are $R - P$ and $Q - P$. The area of the triangle SRP is one half the determinant of the matrix whose rows are $R-P$ and $S-P$. Substituting symbolic co-ordinates for P, Q and R and using $S = aP + bQ + cR$, the result follows upon expanding the determinants.

When a deleted vertex u of degree three is placed inside a face vwx according to 2.1, N equals the total number of faces of G_n corresponding to vwx, $a = f_k(wx)/N$, the proportion of faces corresponding to uwx, and so forth. So the areas of the three triangles created will be proportional to the number of faces of G_n which correspond to them.

The use of these weighted averages to place the points produces a straight-line embedding of G. It improves the layout of the graph because of Lemma 2.4. It still gives an odd shape for the outer face. In section 4 we show how to make the outer face a regular polygon. We first need to define a special family of graphs.

3. A Family of Near Triangulations.

A *near triangulation* is a planar graph all of whose faces are triangles, except possibly for one. We define a family of near triangulations NT_m on $m = \frac{1}{2}(q+1)(q+2)$ vertices, where $q \geq 0$. This is an exceptional family of graphs which must be characterized before the generalization of Read's algorithm is presented. It is a property of planar graphs that any face can be chosen as the outer face in some embedding. We assume that the non-triangular face of NT_m will be the outer face.

3.1 Definition. Let $m = \frac{1}{2}(q+1)(q+2)$, where $q \geq 0$. We define a near triangulation NT_m on m vertices.

1. If $q = 0$ then NT_1 consists of a single vertex and no edges (the degenerate case).
2. Otherwise $q \geq 1$. The outer face of NT_m is a cycle C_{3q} of length $3q$. Let $(a_0, a_1, \ldots, a_{3q-1})$ be the vertices of C_{3q}.
3. If $q = 1$, then $NT_3 = C_3$, a triangle.
4. Otherwise $q \geq 2$. Vertices a_0, a_q, and a_{2q} have degree two. The vertices adjacent to them are joined by edges: $a_{3q-1} \to a_1$, $a_{q-1} \to a_{q+1}$, and $a_{2q-1} \to a_{2q+1}$.
5. If $q = 2$, this defines NT_6, as shown in Fig. 6.
6. Otherwise $q \geq 3$, so that $m \geq 10$. Notice that $m - 3q = \frac{1}{2}(q-1)(q-2)$. We then take a copy of NT_{m-3q} and place it inside the cycle C_{3q}.
7. If $m - 3q = 1$, then $q = 3$ and $m = 10$. We place NT_1, a single vertex, inside C_{3q}, and join it to all vertices of the cycle except for a_0, a_q, and a_{2q}. This gives NT_{10}, shown in Fig. 6.
8. Otherwise $m - 3q > 1$ and the outer face of NT_{m-3q} is a cycle on $3q - 9 = 3p$ vertices, where $p = q - 3$. Let its vertices be $(b_0, b_1, \ldots, b_{3p-1})$. Join b_0 to a_{3q-2}, a_{3q-1}, a_1 and a_2. Join b_p to $a_{q-2}, a_{q-1}, a_{q+1}$ and a_{q+2}. Join b_{2p} to $a_{2q-2}, a_{2q-1}, a_{2q+1}$ and a_{2q+2}. Let $1 \leq i < p$. Vertex b_i is joined to a_{i+1} and a_{i+2}. Vertex b_{p+i} is joined to a_{q+i+1} and a_{q+i+2}. Vertex b_{2p+i} is joined to a_{2q+i+1} and a_{2q+i+2}. This completes the definition of NT_m.

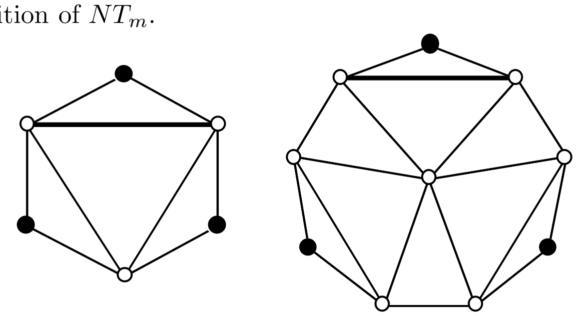

Fig. 6, The graphs NT_6 and NT_{10}.

3.2 Lemma. Let $m = \frac{1}{2}(q+1)(q+2)$, where $q \geq 1$. Let NT_m be as defined above. Then NT_m is a near triangulation, with one face C_{3q} of degree $3q$. Vertices a_0, a_q and a_{2q} have degree two. The other vertices of C_{3q} have degree 4. All remaining vertices of NT_m have degree 6.

Proof. The proof is by induction on q. When $q = 1, 2$, or 3, the lemma is true, as can be seen from Fig. 6. Suppose that it is true up to $q = p \geq 3$ and consider $q = p + 3$. NT_m contains a cycle C_{3q} with vertices $(a_0, a_1, \ldots, a_{3q-1})$. Inside the cycle is a copy of NT_{m-3q}. Since $q \geq 4$, we know that $m - 3q = \frac{1}{2}(q-1)(q-2) = \frac{1}{2}(p+1)(p+2) \geq 3$. So the outer face of NT_{m-3q} is a cycle C_{3p}, with vertices $(b_0, b_1, \ldots, b_{3p-1})$. According to the induction hypothesis, in NT_{m-3q} vertices b_0, b_p and b_{2p} each have degree 2. In NT_m they are also joined to 4 more vertices each, eg., b_p is joined to $a_{q-2}, a_{q-1}, a_{q+1}$ and a_{q+2}. It follows that in NT_m these vertices have degree 6. If $1 \leq i < p$, vertices b_i, b_{p+i} and b_{2p+i} are joined to 2 more vertices each. Thus all vertices of NT_{m-3q} have degree 6 in NT_m.

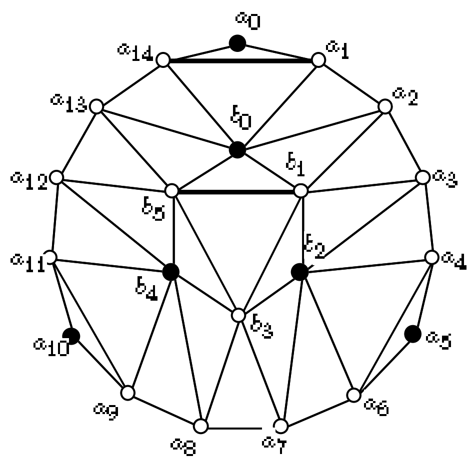

Fig. 7, The graph NT_{21} $(q = 5)$ containing NT_6 $(p = 2)$.

All faces inside NT_{m-3q} are triangles, by the induction hypothesis. The faces between C_{3q} and C_{3p} are also triangles, for the following reason. Vertices $b_1, b_2, \ldots, b_{p-1}$ are each joined to two consecutive vertices of $a_2, a_3, \ldots,$ a_{q-2}. Therefore each a_i is also joined to two consecutive vertices b_{i-1} and b_i. This creates triangles between the two cycles, and also ensures that each a_i has degree 4, except for a_0, a_q and a_{2q}, which have degree two. This completes the proof of the lemma.

Given $m = \frac{1}{2}(q+1)(q+2)$, where $q > 0$, it is easy to verify that NT_m consists of $1 + \lfloor \frac{q}{3} \rfloor$ concentric "shells", where the innermost shell is a single vertex if $q \equiv 0 \pmod 3$. The outer shell is C_{3q} with vertices

$(a_0, a_1, \ldots, a_{3q-1})$. Figs. 6 and 7 show planar embeddings of NT_6, NT_10 and NT_21 in which the vertices of C_{3q} are equally spaced along the circumference of a circle.

4. The Outer Face.

In this section we show how to make the outer face a regular polygon. Let G be given. We assume that G is a 2-connected graph. (A graph which is not 2-connected can be converted to a 2-connected graph by the addition of some edges, which can later be removed once an embedding has been constructed.) In order to select the outer face, we scan through all faces of G and pick one of largest degree. Any face could be used, but it is convenient to choose a face of largest degree. Let F be the the boundary of this face, that is, F is a subgraph of G isomorphic to a cycle. Add a new vertex x_0 to G, adjacent to every vertex of F. This triangulates the face F. Call the resulting graph G^+. Now triangulate G^+ by adding diagonals to non-triangular faces. This gives a triangulation which we will still call G_n, although it now has $n+1$ vertices. $V(F)$ is the set of vertices adjacent to x_0 in G_n. This is illustrated in Fig. 8.

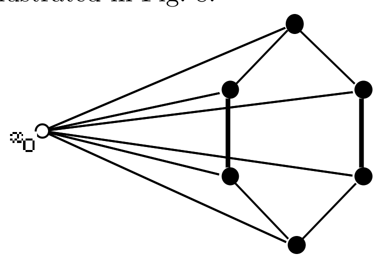

Fig. 8, Triangulating the face with boundary F.

We now reduce G_n through G_{n-1}, G_{n-2}, G_{n-3}, ... The reduction of Read's algorithm is modified somewhat. We first select three vertices $u_0, v_0, w_0 \in V(F)$ which are approximately equally spaced with respect to the cycle F. Write $V_0 = \{x_0, u_0, v_0, w_0\}$. The vertices of V_0 will not be deleted during the reduction. The positions of the vertices of F are now precomputed. We select a circle in the region available for drawing G, and distribute the vertices of F evenly along the circle. This defines $P(v)$ for each $v \in V(F)$. When joined by straight lines, the vertices of F will form a convex regular polygon in the final embedding. The vertices of G_n are then classified according to their degree.

Now let G_k be given, where initially $k = n$. A vertex u_k to be deleted is selected so that $u_k \notin V(F)$, whenever this is possible. Each G_k will contain a cycle F_k consisting of the vertices adjacent to x_0 such that $V(F_k) \subseteq V(F)$. The remaining vertices of G_k are $R_k = V(G_k) - V(F_k) - x_0$. The initial situation is $F_n = F$.

4.1. Choose u_k according to the following rules.

1. If G_k contains a vertex $u \notin V_0$ of degree 3 such that $u \in R_k$, then $u_k = u$.
2. Else if G_k contains a vertex $u \notin V_0$ of degree 3 such that $u \in F_k$, then $u_k = u$.
3. Else if G_k contains a vertex $u \notin V_0$ of degree 4 such that $u \in R_k$, then $u_k = u$.
4. Else if G_k contains a vertex $u \notin V_0$ of degree 5 such that $u \in R_k$, then $u_k = u$.
5. Else if G_k contains a vertex $u \notin V_0$ of degree 4 such that $u \in F_k$, then $u_k = u$.
6. If a vertex u_k was found, then reduce G_k to G_{k-1} by deleting u_k as in Read's algorithm, as shown in Figs. 1, 2, and 3, with an additional restriction: if $u_k \in F_k$ and $\deg(u_k) = 4$, let v and x be the vertices adjacent to u_k in the cycle F_k. Then vx is the diagonal added to $G_k - u_k$. This defines $F_{k-1} = F_k - u_k + vx$.
7. Set $k := k - 1$ and repeat from step 1 until either $R_k = \emptyset$ or no vertex u_k satisfying the conditions was found.

Step 4.1.6 ensures that $F_{k-1} = F_k - u_k + vw$ is a cycle in G_{k-1}. Notice that G_k is always a triangulation on $k + 1$ vertices, and that every G_k contains the vertices V_0. If G_3 is reached, it is a tetrahedron on the vertices x_0, u_0, v_0, w_0.

4.2 Theorem. *Let G_n be reduced by steps 4.1.1 to 4.1.7 above, through $G_{n-1}, G_{n-2}, \ldots, G_{k_0}$, where $k_0 \geq 3$. Then each G_k is a well-defined simple triangulation, for $k = n, n-1, \ldots, k_0$. For each value of k, F_k is a cycle in G_k consisting of those vertices adjacent to x_0. Each vertex of G_k has degree at least three.*

Proof. The proof is by reverse induction on k. When $k = n$ it is clearly true. Let u_k be the vertex chosen to be deleted, according to steps 4.1.1 to 4.1.6 above. The vertices of R_k will eventually be embedded in the interior of F. There are several cases to consider.

Case 1. $\deg(u_k) = 3$ (steps 4.1.1 and 4.1.2). If $G_k - V_0$ contains a vertex of degree three, it will be the first selected to be deleted. If $u_k \in R_k$ then the reduction is exactly as in Fig. 1. $G_{k-1} = G_k - u_k$ is a triangulation and $F_{k-1} = F_k$ is the cycle adjacent to x_0. If $u_k \in F_k$, let v and w be the vertices of F_k adjacent to u_k. Then $v \to w$ since G_k is a triangulation, so that $F_{k-1} = F_k - u_k + vw$ is the cycle adjacent to x_0. The degrees of v and w will still be at least three.

Case 2. $\deg(u_k) = 4$, where $u_k \in R_k$ (step 4.1.3). This case can occur only if $V(G_k) - V_0$ has no vertex of degree three. The reduction is exactly as in Fig. 2 where a diagonal edge vw is added. $G_{k-1} = G_k - u_k + vw$

is a triangulation and $F_{k-1} = F_k$ is the cycle adjacent to x_0. Note that by 1.1, two vertices have their degree decreased by one. Every vertex of $V(G_k) - V_0$ has degree at least 4, so their degrees can decrease to 3, but not to 2. The vertices of V_0 are on the cycles F_k and F_{k-1}, and they are adjacent to x_0. Therefore they always have degree at least three.

Case 3. $\deg(u_k) = 5$, where $u_k \in R_k$ (step 4.1.4). This case can occur only if $V(G_k) - V_0$ has no vertex of degree three and no vertex of degree 4 not on F_k. The reduction is exactly as in Fig. 3. Two diagonal edges vx and vy are added. $G_{k-1} = G_k - u_k + vx + vy$ is a triangulation and $F_{k-1} = F_k$ is the cycle adjacent to x_0. By 1.1, two vertices w and z have their degree decreased by one. If $w, z \in R_k$, then they had degree at least 5. So they now have degree at least 4. If one or both of them are on F_k their degree could be as small as 4. So their degree would now be at least 3.

Case 4. $\deg(u_k) = 4$, where $u_k \in F_k$ (step 4.1.5). This case can occur only if $V(G_k) - V_0$ has no vertex of degree three and R_k has no vertex of degree 4 or 5. So every vertex of R_k has degree at least 6. Let the vertices adjacent to u_k be v, w, x and x_0 where v and x are the vertices adjacent to u_k on F_k, and $w \in R_k$. This is illustrated in Fig. 9.

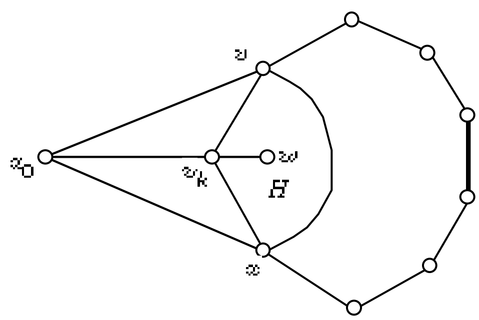

Fig. 9, Vertex $u_k \in F_k$, $\deg(u_k) = 4$.

If $v \not\to x$ then we can delete u_k and add the diagonal vx to get G_{k-1} and F_{k-1}, with the required properties. So suppose that $v \to x$. The edge vx is a chord of the cycle F_k creating a triangle vxu_k. Let us delete all vertices of G_k except those on or inside the triangle vxu_k to get a graph H. Since G_k is a triangulation, so is H. Every vertex of H has degree at least 6, except for v, x, and u_k. But if $n_3, n_4, n_5, \ldots$ are the number of vertices of H of degree 3, 4, 5, etc, then by 1.2, $3n_3 + 2n_4 + n_5 = 12 + n_7 + 2n_8 + \ldots$. This equation cannot be satisfied when all but three vertices of H have degree 6 or more. Therefore $v \not\to x$ and G_{k-1} and F_{k-1} with the required properties can always be formed, that is, the requirement of step 4.1.6 can always be adhered to.

It follows that the graphs $G_n, G_{n-1}, \ldots, G_{k_0}$ all have the required properties. The reduction process continues until either $R_k = \emptyset$, or until no vertex u_k of the desired type can be found. G_{k_0} is the last graph in the sequence. If $R_{k_0} = \emptyset$, then the reinsertion process can be executed, since $V(F_{k_0}) \subseteq V(F)$ and the positions of the vertices of F have been precomputed. The vertex x_0 is not embedded. Its only purpose is to ensure that G_n is a triangulation and that the vertices of each F_k have degree at least three. It is possible that $R_{k_0} \neq \emptyset$, and that no vertex u_k can be selected according to the criteria of 4.1.1 to 4.1.5. There is a unique family of graphs for which this occurs. They are the near triangulations NT_m constructed in section 3. This is characterized in the following sequence of lemmas.

4.3 Lemma. Suppose that $R_k \neq \emptyset$ and that G_k contains no vertex u_k meeting the criteria of 4.1.1 to 4.1.5. Then

(i) every vertex of R_k has degree 6;

(ii) u_0, v_0, and w_0 all have degree 3;

(iii) every vertex of $F_k - V_0$ has degree 5.

Proof. Let ℓ denote the length of F_k, ie, $\ell = |V(F_k)|$. Let F_k contain $m_3, m_4, m_5, \ldots$ vertices of degree 3, 4, 5, etc. Let R_k contain $n_3, n_4, n_5, \ldots$ vertices of degree 3, 4, 5, etc. Vertex x_0 has degree ℓ. Let N and ε denote the number of vertices and edges of G_k, respectively. Note that G_k contains no vertices of degree 2 or less. Then $\sum_{i\geq 3} m_i = \ell$, and $\sum_{i \geq 3} n_i = N - \ell - 1$. Each vertex of F_k is adjacent to x_0 and to two other vertices of F_k, since F_k forms a cycle. Let C denote the number of chords of F_k, and let X denote the number of edges connecting F_k to R_k. Then

$$\sum_{i\geq 3} i m_i = 3\ell + 2C + X$$

and

$$\sum_{i\geq 3} i m_i + \sum_{i\geq 3} i n_i = 2\varepsilon - \ell$$

From 1.2 we have $6N - 2\varepsilon = 12$. Into this we substitute $N = 1 + \ell + \sum_{i\geq 3} n_i$ and $2\varepsilon = 4\ell + 2C + X + \sum_{i\geq 3} i n_i$, and simplify, to obtain

4.4. $\quad 3n_3 + 2n_4 + n_5 = 6 + 2C + X - 2\ell + n_7 + 2n_8 + 3n_9 + \ldots$

Every vertex of F_k, except for $\{u_0, v_0, w_0\}$, has degree at least 5, for otherwise 4.1.2 or 4.1.5 would apply. Its adjacencies to x_0 and F_k account for 3 incident edges. Therefore each contributes at least 2 edges to the sum $2C + X$. There are $\ell - 3$ such vertices, so that $2C + X \geq 2\ell - 6$. If $2C + X > 2\ell - 6$, then the right hand side of 4.4 is strictly positive, so that R_k must have a vertex of degree 3, 4, or 5. So suppose that $2C + X = 2\ell - 6$.

Notice that equality is possible only if vertices u_0, v_0, and w_0 all have degree exactly equal to 3, and every vertex of $V(F_k) - V_0$ has degree exactly equal to 5. This gives conditions (ii) and (iii) of the lemma. Substituting $2C + X = 2\ell - 6$ into 4.4 gives $3n_3 + 2n_4 + n_5 = n_7 + 2n_8 + 3n_9 + \ldots$. If G_k contains no suitable vertex u_k, then by 4.1.1 to 4.1.5, R_k contains no vertex of degree 3, 4, or 5, so that $3n_3 + 2n_4 + n_5 = 0$. This is possible only if $n_3 = n_4 = n_5 = n_7 = n_8 = \ldots = 0$. So every vertex of R_k has degree 6, which is statement (i) above.

The conditions (i), (ii) and (iii) are very restrictive for G_k, since it is a triangulation. The cycle F_k has three vertices u_0, v_0, w_0 of degree three, and all other vertices of degree five. Since F_k is a cycle, the vertices of F_k have a natural cyclic ordering. For any $a \in V(F_k)$, let a^- and a^+ respectively denote the vertices before and after a in the cyclic order of F_k. Since G_k is a triangulation and $\deg(u_0) = 3$, it follows that $u_0^+ u_0^-$ is a chord of F_k. Similarly for $v_0^+ v_0^-$ and $w_0^+ w_0^-$. Consequently $\ell \geq 6$ and F_k has $C \geq 3$ chords. This is illustrated in Fig. 10. Vertex x_0 is not shown in the diagram. It is understood that F_k is the outer face of $G_k - x_0$.

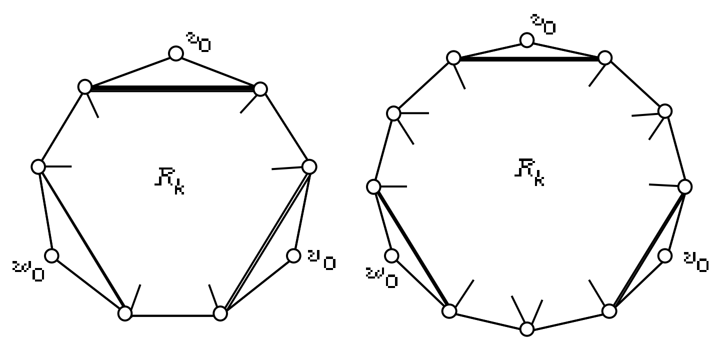

Fig. 10, The cycle F_k, $\ell = 9$ and $\ell = 12$.

If ab is any chord of F_k, then it divides F_k into two paths, the vertices from a^+ up to b, and those from b^+ up to a. Denote these two paths by $[a^+, b]$ and $[b^+, a]$.

4.5 Lemma. Let G_k be as in 4.3. Then the cycle F_k has exactly three chords, that is, $C = 3$.

Proof. Suppose that ab were a chord of F_k other than $u_0^+ u_0^-$, $v_0^+ v_0^-$, and $w_0^+ w_0^-$. Without loss of generality, choose ab so that one of the paths $[a^+, b]$ and $[b^+, a]$ has minimum length, say it is $[a^+, b]$. Refer to Fig. 11. Since G_k is a triangulation, edge ab is contained in two triangles. There cannot be an edge $a^+ b$ or ab^- since both of these would be chords with a shorter path than $[a^+, b]$. Therefore there is a vertex $c \in R_k$ such that abc forms a triangle on the same side of ab as a^+. Since $\deg(a) \leq 5$ and $\deg(b) \leq 5$, there can be no more edges incident at a or b. But there must be another triangle containing ab, on the other side of the edge. This is a contradiction. Therefore $u_0^+u_0^-$, $v_0^+v_0^-$, and $w_0^+w_0^-$ are the only chords of F_k.

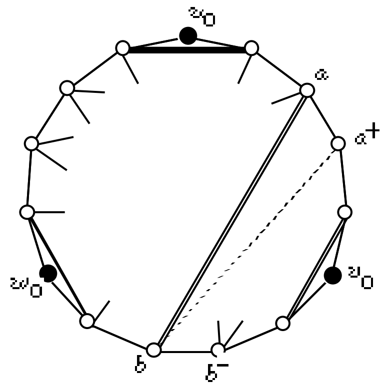

Fig. 11, The cycle F_k with chord ab.

As a consequence of 4.5 we notice that if $\ell \geq 9$ the vertices $\{u_0^+, v_0^+, w_0^+, u_0^-, v_0^-, w_0^-\}$ are all joined to R_k by exactly one edge. Call these *type I* vertices. The remaining vertices of F_k (excluding u_0, v_0, w_0) are joined to R_k by exactly two edges. Call these *type II* vertices.

We show that $G_k - x_0$ must be one of the graphs NT_m. By Lemmas 4.3 and 4.5, every vertex of R_k has degree 6, and F_k has exactly three chords. It is clear that NT_m is a graph with these properties. We show that these are the only graphs satisfying these restrictions. We will need the following obsevation, which is an immediate consequence of 1.2.

4.6. *Let $U \subseteq R_k$ and let X_U denote the number of edges with one end in U. Then:*

(i) if $|U| = 1$, then $X_U = 6$;
(ii) if $|U| = 2$, then $X_U \geq 10$;
(iii) if $|U| \geq 3$, then $X_U \geq 12$.

4.7 Lemma. *Let $a \in R_k$ and let $u \in F_k$ be a type II vertex. Then a is not adjacent to both u^+ and u^-.*

Proof. If $a \to u^+, u^-$ then the cyle (a, u^-, u, u^+) encloses one or more faces of G_k. Now u is a type II vertex, which can be joined to a by at most one edge, since G_k is simple. Therefore there must be at least one vertex of R_k inside the cycle (a, u^-, u, u^+). So let U be the vertices of R_k inside the cycle. The number of edges from u, u^+ and u^- to U is at most 4. There are at most 4 edges from a to U. Therefore $X_U \leq 8$. By 4.6 this requires that $|U| = 1$, which is not possible since G_k is a simple graph.

4.8 Theorem. Suppose that the reduction stops with G_{k_0}, where $R_{k_0} \neq \emptyset$. Then $G_{k_0} - x_0$ is isomorphic to NT_m for some $m \geq 10$.
Proof. Write $k = k_0$ and let ℓ be the length of F_k. If $\ell < 9$ then one of $\{u_0^+, v_0^+, w_0^+\}$ is in the set $\{u_0^-, v_0^-, w_0^-\}$. Without loss of generality, suppose that $u_0^+ = v_0^-$. The chords $u_0^- u_0^+$ and $v_0^- v_0^+$ must be contained in a triangle. This implies that $u_0^- v_0^+$ is a chord of F_k, which requires that $w_0^+ = u_0^-$ and $v_0^+ = w_0^-$ so that $\ell = 6$ and $R_k = \emptyset$. Therefore we may assume that $\ell \geq 9$.

Consider vertex u_0^+, which has degree 5. It is adjacent to x_0 and to three vertices of F_k. Since F_k has only three chords, and since $\ell \geq 9$, we can assume without loss of generality that $u_0^+ \to a \in R_k$. The chord $u_0^+ u_0^-$ must be contained in two triangles. Since u_0^+ has degree 5, it must be that $u_0^- \to a$, too. This accounts for all 5 edges at u_0^+ and u_0^-. But the edge $a u_0^+$ must also be contained in two triangles. Therefore $a \to u_0^{++}$. Similarly $a \to u_0^{--}$. See Fig. 12. Thus we have shown:

4.9. If $u_0^{\pm} \to a \in R_k$ then $a \to u_0^+, u_0^{++}, u_0^-, u_0^{--}$.

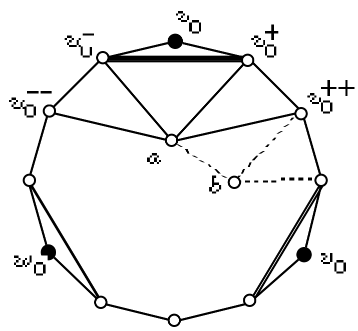

Fig. 12, $u_0^+ \to a \in R_k$ implies that $a \to u_0^+, u_0^{++}, u_0^-, u_0^{--}$

It is possible that $u_0^{++} = v_0^-$. In this case, $a \to v_0^+, v_0^{++}$, as well. This makes $\deg(a) = 6$. If $\ell = 9$ then $v_0^{++} = w_0^-$ and $w_0^{++} = u_0^-$. This accounts for all edges incident with F_k, and so defines $G_k - x_0 = NT_{10}$. See Fig. 6. Otherwise $\ell > 9$. In the reduction algorithm 4.1, vertices u_0, v_0 and w_0 were chosen to be evenly spaced along the cycle F_k. Consequently $\ell \leq 11$. The cycle $(a, v_0^{++}, \ldots, w_0^-, w_0^+, \ldots, u_0^{--})$ has at most 7 vertices, which are adjacent to R_k by at most 8 edges. By 4.6 there can be at most one vertex of R_k inside this cycle. But this is impossible, since G_k is a simple graph. It follows that if $u_0^{++} = v_0^-$, then $G_k - x_0 = NT_{10}$ is the only possibility. We may therefore assume that $\ell \geq 12$.

Consider the edge $a u_0^{++}$. u_0^{++} is a type II vertex. By Lemma 4.7, $a \not\to u_0^{+++}$. Since $a u_0^{++}$ is contained in two triangles, there must be a vertex $b \in R_k$ such that $b \to a, u_0^{++}$. This makes $\deg(u_0^{++}) = 5$. The edge $b u_0^{++}$ must also be contained in two triangles, so that we conclude that $b \to u_0^{+++}$. At this point there are two possibilities. Either u_0^{+++} is a type II vertex, or else $u_0^{+++} = v_0^-$. In the former case we invoke 4.6 applied to vertex b, and conclude that R_k contains a vertex $c \to b, u_0^{+++}$. Notice that b is adjacent to exactly two consecutive vertices of F_k. We then apply the same argument to c. We get a path $(a, b, c \ldots)$ in R_k whose vertices are joined to two consecutive vertices of F_k, until a vertex $d \to v_0^-$ is reached. By 4.9, $d \to v_0^+, v_0^{++}, v_0^-, v_0^{--}$. We then continue the argument from v_0^{++} until w_0^- is reached, at which point we again invoke 4.9. The final result is a cycle $C = (a_0, a_1, a_2, \ldots, a_{\ell-9})$ in R_k with three vertices a_0, b_0 and c_0 each adjacent to 4 consecutive vertices of F_k. The remaining vertices are each adjacent to two consecutive vertices of F_k. The vertices a_0, b_0 and c_0 are evenly spaced along C. If we now take the subgraph of G_k induced by R_k and attach a vertex y_0 adjacent to every vertex of C, we get a graph R_k^+ with exactly the same properties as G_k, but with fewer vertices. We use induction and conclude that $R_k = NT_p$ for some p. But then by the recursive definition of NT_p we conclude that $G_k - x_0 = NT_m$, where $m = \frac{1}{2}(\frac{\ell}{3}+1)(\frac{\ell}{3}+2)$. This completes the proof of the theorem.

The reduction process 4.1 reaches a graph G_{k_0} where either $R_{k_0} = \emptyset$ or else there is no suitable vertex u_k to delete, in which case $G_k - x_0 = NT_m$ for some $m \geq 10$. We then continue the reduction as follows.

4.10. If 4.1 terminates with $R_k \neq \emptyset$, change the set V_0 to $\{x_0, u_0^+, v_0^+, w_0^+\}$. Execute 4.1 again, until $R_k = \emptyset$.

4.11 Lemma. The reduction process 4.1 augmented by 4.10 always terminates with $R_k = \emptyset$.

Proof. Vertices u_0, v_0 and w_0 have degree three, so they can be deleted by 4.1.2 once 4.10 has been executed. Since u_0, v_0 and w_0 are equally spaced along F_k, so are u_0^+, v_0^+ and w_0^+. Consequently if 4.1 again stops with $R_k \neq \emptyset$, G_k will again be a near triangulation NT_m, for some m, so that 4.10 can again be executed, until eventually $R_k = \emptyset$ is reached.

Thus we always have $R_{k_0} = \emptyset$. The positions of the vertices of F have all been previously computed. Their precomputed values will not be changed. The vertex x_0 is not embedded. We begin by embedding the vertices of F_{k_0} according to their precomputed positions. We are now in a position to reinsert the deleted vertices. Suppose that G_{k-1} has just been embedded and that we are about to reinsert the deleted vertex u_k to get G_k.

4.12. Reinsert vertex u_k as follows.

1. If $u_k \in F_k$, we assign u_k its precomputed position $P(u_k)$.
2. If $u_k \in R_k$, we compute $P(u_k)$ as a weighted average according to 2.1.
3. Set $k := k + 1$ and repeat until $k = n$ is reached.

4.13 Theorem. *Let F be any face of G and let $G_n, G_{n-1}, \ldots, G_{k_0}$ be constructed as above. The reinsertion algorithm produces a straight line planar embedding of G such that the vertices of F lie on a regular polygon.*

Proof. First, it is clear the vertices of F lie on a regular polygon. When G_{k_0} is reached, a straight-line embedding planar of it is constructed. We must show that when vertex u_k is re-inserted in G_{k-1}, that the resulting embdding of G_k is a planar embedding. If $u_k \in R_k$, then u_k is placed inside a triangle of G_{k-1}, according to 2.2. Read's algorithm [5] guarantees that the resulting G_k will be a straight line planar embedding. If $u_k \in F_k$, then $P(u_k)$ has been previously computed. When u_k was deleted, it had degree 3 or 4. If $\deg(u_k) = 3$, it is adjacent only to vertices of F_k and x_0. Since x_0 is not embedded, and the vertices of F_k lie on a circle, G_k will be a planar embedding in this case. If $\deg(u_k) = 4$, then u_k is adjacent to only one vertex of R_k. This is the third vertex of the triangle containing the vertices of F_k adjacent to u_k. Clearly no crossing can be introduced by joining u_k to this vertex. So G_k is a straight-line planar embedding in all cases. This completes the proof of the theorem.

The embeddings produced by this algorithm for the graphs of Figs. 4, 5, and 6 are shown below. This is the basic algorithm used by the "Groups & Graphs"* software [3], version 2.3, to produce planar layouts. It is also used to "pivot" a given embedding of a planar graph by selecting an arbitrary face F as the outer face.

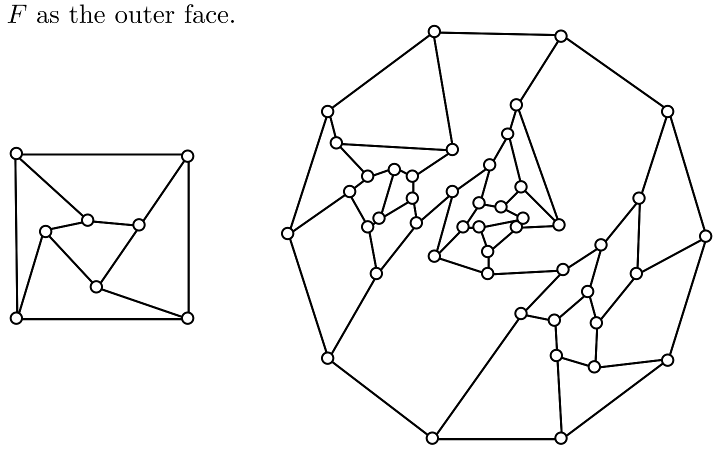

Fig. 13, Embeddings of the cube and Tutte's graph.

The algorithm has linear complexity. The degrees of the vertices are computed. This requires $O(n)$ steps. They are in the range $2, 3, \ldots, n-1$,

* Available on the internet via anonymous ftp from ftp://ftp.cc.umanitoba.ca/pub/mac, or from http://130.179.24.217/G&G/G&G.html

so that they can be sorted in linear time. The degrees of the faces are also computed, which again takes linear time (see [4]). A face F is selected as the outer face, and the coordinates of its vertices are computed. The remaining faces of G are triangulated, which again takes linear time, by 1.2. There are a number of iterations in which a vertex u_k to be deleted is selected. Each deletion takes at most a constant number of steps, since $\deg(u_k) \leq 5$. Each time 4.10 is executed, only a constant number of steps are required. It is executed a linear number of times. When G_{k_0} is reached, it is embedded in linear time. The re-insertion process again takes linear time. So the entire algorithm is $O(n)$. In practice, using the Groups & Graphs software, embeddings of graphs are produced almost instantaneously, up to around 100 vertices or more.

A number of modifications of this basic algorithm are possible. For example, it was found that with the linearly weighted averages of 2.1, long thin triangles and smaller, more compact triangles of equal area are given equal preference, because of 2.4. We found that in practice, it was convenient to modify the linear weights slightly to reduce the tendency to produce long thin triangles. We also found it suitable to modify 2.2 for vertices of degree 4, by allowing the reinserted vertex to move off the edge connecting v to x. Another modification that we have experimented with is this. Once G has been embedded, use the method of Eades [2] which places "springs" on the edges, and allow the vertices to "vibrate" for several iterations. This has the effect of spreading the distribution of vertices more evenly on the plane, and reducing the average edge-length. However it is non-linear, and tends to introduce crossings if one is not careful.

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