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Balanced Network Flows

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Abstract

Let G be a simple, undirected graph. A special network N, called a balanced network, is constructed from G such that maximum matchings and f-factors in G correspond to maximum flows in N. A max-balanced-flow-min-balanced-cut theorem is proved for balanced networks. It is shown that Tutte's Factor Theorem is equivalent to this network flow theorem, and that f-barriers are equivalent to minimum balanced edge-cuts. A max-balanced-flow algorithm will solve the factor problem.

1. Balanced Networks.

Let G be a simple graph, directed or undirected, with vertex set V(G) and edge set E(G). A *network* N is a directed graph which contains two special vertices s and t, the *source* and *target*, respectively, and in which every edge e is assigned a positive integer-valued *capacity* cap(e). Terminology for graphs, networks, and flows is taken from Bondy and Murty [1]. Edges of a directed graph are ordered pairs of vertices. If (u,v) is an edge, we indicate that u is adjacent to v by $u \rightarrow v$. Edges of an undirected graph are unordered pairs, and we write the pair {u,v} as uv, where the order is unimportant. In a directed graph we also often use uv to indicate one of the edges (u,v) or (v,u), especially if the direction is not explicitly given. The opposite direction will then be given by vu. Let f be an integer-valued function on E(N). Given any u V(N), the *out-flow* at u is $f^+(u) = \sum_{v,\, u \rightarrow v} f(uv)$ and the *in-flow* at u is $f^-(u) = \sum_{v,\, v \rightarrow u} f(vu)$. The function f is called a *flow* if it satisfies the two conditions:

i) $f^+(u) = f^-(u)$, for all vertices $u \neq s,t$, (the conservation condition)

ii) $0 \leq f(uv) \leq cap(uv)$, for all edges uv (the capacity constraint).

The *value* of the flow is $val(f) = f^+(s) - f^-(s)$, that is, the *net* out-flow at the source s. We are interested in a special kind of network, called a balanced network.

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1.1 Definition. A *balanced network* is a network N with the following properties:

- The vertices of N consist of a source s, a target t, and two sets of vertices $X=\{x_1,x_2,...,x_n\}$ and $Y=\{y_1,y_2,...,y_n\}$;
- N contains the pair of edges (s,x_i) and (y_i,t) for $1 \le i \le n$;
- The remaining edges of N are between X and Y, and occur in pairs, either (x_i,y_j) and (x_j,y_i), or (y_i,x_j) and (y_j,x_i), where $i \ne j$;
- $cap(sx_i)=cap(y_it)$, $cap(x_iy_j)=cap(x_jy_i)$, and $cap(y_ix_j)=cap(y_jx_i)$, for all i,j.

The vertices of N occur in complementary pairs. Given any vertex u V(N), the complementary vertex will be denoted by u′. Thus $s'=t$, $t'=s$, $x_i' = y_i$, and $y_i' = x_i$. The edges also occur in complementary pairs: $(x_i,y_j)'=(x_j,y_i)$ and $(s,x_i)'=(y_i,t)$. Notice that $(u,v)'=(v',u')$. Complementary edges always have the same capacity, $cap(uv)=cap(v'u')$. A *balanced flow* in N is a flow f in which every complementary pair of edges carries the same flow, that is, $f(uv)=f(v'u')$. We want to find a maximum balanced flow in N. As can be seen from the following example, a balanced network is bipartite. The network in Fig. 1 has $cap(x_iy_j)=1$ and $cap(sx_i)=cap(y_it)=d_i$.

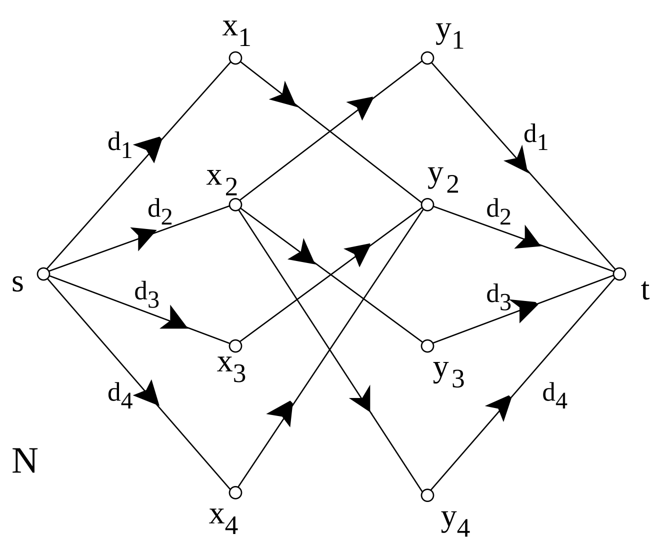

Fig 1. A balanced network. Edge capacities are d_1, d_2, d_3, d_4 or 1.

Before proceeding further, we explain the motivation for introducing balanced networks and balanced flows. Given a graph G, consider the following problems.

Max Matching: Find a maximum matching in G;
Factor Problem: Given an integer-valued function b on V(G), does G have a subgraph H such that $deg_H(u) = b(u)$, for all u V ?

If G were a bipartite graph, we could solve these problems with the standard techniques of flow theory. Consider the factor problem. Let the bipartition of G be (X,Y). Create a network N from G as follows. Direct the edges of G from X to Y and assign them a capacity of 1. Add a source s and target t, plus all edges (s,x) and (y,t), for all x X and y Y. Assign capacities b(x) and b(y) to sx and yt, respectively. Find a maximum flow f in N. It is easy to see that the subgraph H exists if and only if the value of f is

$$\mathrm{val}(f) = \sum_{x \quad X} b(x) = \sum_{y \quad Y} b(y).$$

To solve the maximum matching problem, just set b(u)=1 for all vertices u, and find a maximum flow.

When G is not bipartite, standard flow theory does not apply. However, Tutte's Theorem [1,6,10] gives necessary and sufficient conditions for a perfect matching to exist and can be used to obtain a formula for the size of a maximum matching. Tutte's Factor Theorem [6,8,9,10] solves the second problem. In [10], Tutte shows that finding an f-factor in G is equivalent to finding a 1-factor in a considerably larger graph formed by transforming G.

The purpose of balanced flows is to show that these problems can in fact be solved by the methods of flow theory. It is shown that Tutte's factor theorem is equivalent to a min-max flow theorem, the *Max-Balanced-Flow-Min-Balanced-Cut Theorem*. In terms of network flows, an f-barrier is equivalent to a minimum balanced edge-cut. These results are simpler to derive than the standard structure theory of f-factors (cf. Lovasz and Plummer [6, section 10.2]), since the methods of network flows and edge-cuts apply. In [7] Schrijver states that combinatorial min-max relations often yield elegant theorems and are closely related to the algorithmic solution to a problem. The corresponding algorithm here is the *max-balanced-flow* algorithm [4]. This algorithm can also be used to solve the capacitated b-matching problem (see [7]).

2. Balanced Edge-Cuts.

Let N be a network. Let S V(N) and let $\overline{S}$ denote V(N)–S. The *edge-cut* K=[S,$\overline{S}$] consists of all edges uv such that u S and v $\overline{S}$. Let $\overline{K}$ denote [$\overline{S}$,S]. Write

$$f^+(S) = \sum_{uv \quad K} f(uv) \qquad \text{and} \qquad f^-(S) = \sum_{uv \quad \overline{K}} f(uv).$$

By the conservation condition $f^+(S)=f^-(S)$ unless s or t is in S. We shall assume that s S and t $\overline{S}$, unless otherwise indicated. Then $\mathrm{val}(f)=f^+(S)-f^-(S)$, for all such S. The capacity of the edge-cut is cap(K) and the total flow on K is f(K), where

$$\mathrm{cap}(K) = \sum_{uv \quad K} \mathrm{cap}(uv) \qquad \text{and} \qquad f(K) = \sum_{uv \quad K} f(uv) = f^+(S)$$

By standard flow theory, val(f)≤cap(K) for all flows f and all edge-cuts K. The *Max-Flow-Min-Cut Theorem* states that in any network, the value of a maximum flow f equals the value of a minimum edge-cut K. Furthermore, the flow f satisfies f(uv)=cap(uv) for all uv K, and f(uv)=0 for all uv $\overline{K}$.

These results also hold for arbitrary flows and arbitrary edge-cuts in a balanced network, of course. However since we want a balanced flow in N, we must restrict attention to certain special kinds of edge-cuts.

Given an arbitrary edge cut $K=[S,\bar{S}]$ in a balanced network N, we divide the vertices of X and Y into 4 sets:

$$A = \{x_i, y_i \mid x_i \;\; \bar{S},\ y_i \;\; S\}$$
$$B = \{x_i, y_i \mid x_i \;\; S,\ y_i \;\; \bar{S}\}$$
$$C = \{x_i, y_i \mid x_i \;\; S,\ y_i \;\; S\}$$
$$D = \{x_i, y_i \mid x_i \;\; \bar{S},\ y_i \;\; \bar{S}\}$$

A may then be further subdivided into $A_x = A \cap X$, and $A_y = A \cap Y$, and the other 3 sets may also be subdivided in the same manner. This is illustrated in Fig. 2.

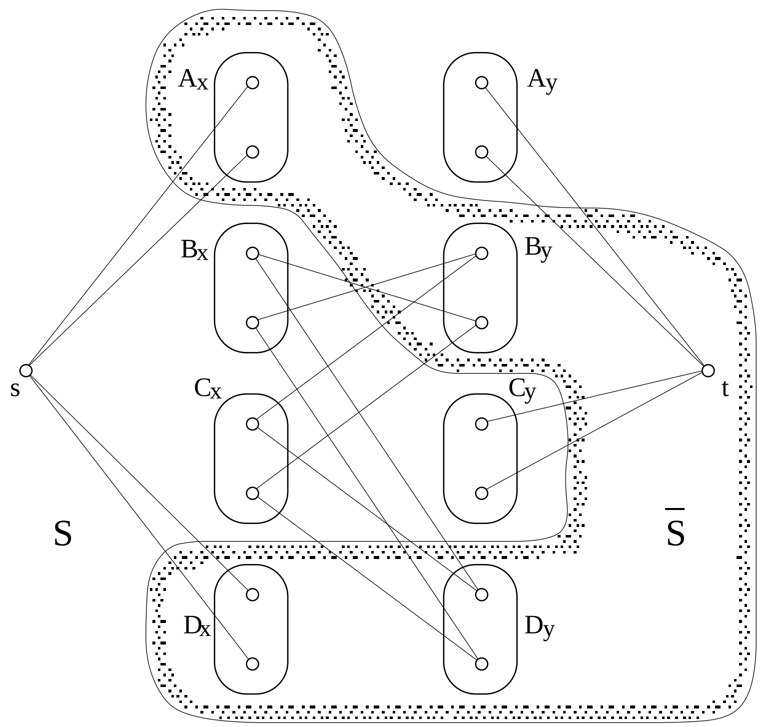

Fig. 2, An edge-cut K. Edges illustrated belong to K.

We know that val(f)≤cap(K) in general, but for balanced networks, this statement is not as strong as it could be. With arbitrary networks one can always find a flow f such that val(f)=cap(K) for some edge cut K, by the method of augmenting paths. But with balanced networks, it is not always possible to find a balanced flow f such that val(f)=cap(K). It would therefore be useful to find a special kind of edge cut K in a balanced network N, such that K has properties in N that are analogous to those of the usual kind of edge-cut in an arbitrary network.

The subgraph of N induced by C is denoted N[C]. If C≠Ø, N[C] will be a graph with one or more connected components. Let C_1, C_2, …, C_k be its components. Let K_i denote those edges of K with one endpoint in C_i, for i=1,2,…,k, and let K_r denote the remaining edges of K, that is, those with no endpoint in C. We also define $\bar{K}_i$ to be those edges of $\bar{K}$ with one endpoint in C_i, and $\overline{K}_r$ to be the remaining edges of $\overline{K}$.

2.1 Definition. A *balanced edge-cut* in N is an edge-cut K that satisfies the following conditions:

i) There are no edges between C and D;

ii) If u C_i, then u′ C_i ;

iii) if C≠Ø, then cap(K_i) is odd, for $1 \le i \le k$.

It will be seen that there is always an edge-cut of this type corresponding to a maximum balanced flow. The following notation is now required. Any given component C_i will be denoted by I. Thus I_x denotes $C_i \cap C_x$ and I_y denotes $C_i \cap C_y$. Let E_{AB} denote all edges of N directed from A to B. It can be divided into those edges directed from A_x to B_y and those directed from A_y to B_x . In most applications, N will contain no edges directed from Y to X at all, but it is easy to include them in the following proofs. Denote the former set by E^+_{AB} and the latter by E^-_{AB}. Then $E_{AB}=E^+_{AB}$ E^-_{AB}. (In most cases $E^-_{AB}=Ø$.) Write $X_{AB}=\text{cap}(E_{AB})$ and $F_{AB}=f(E_{AB})$. Then as above we can write $X_{AB} = X^+_{AB} + X^-_{AB}$ and $F_{AB} = F^+_{AB} + F^-_{AB}$. By complementarity, $F^+_{AB} = F^+_{BA}$ and $F^-_{AB} = F^-_{BA}$. If E_{sA} denotes all edges directed from s to A, then X_{sA} denotes cap(E_{sA}) and F_{sA} denotes f(E_{sA}). This notation is extended to the sets B,C,D, and I in the obvious way.

2.2 Lemma. Let K be a balanced edge-cut. For any component C_i,

$$(F^+_{II} - F^-_{II}) + (F^+_{IA} - F^-_{IA}) + (F^+_{IB} - F^-_{IB}) - F_{sI} = 0.$$

Proof. Since the C_i are connected components of N[C] and there are no edges from C to D, the only edges directed out of I_x are those from I_x to I_y, A_y, and B_y. Hence $f^+(I_x) = F^+_{II} + F^+_{IA} + F^+_{IB}$. The only edges directed in to I_x are those from I_y, A_y, B_y, and s. Hence $f^-(I_x) = F^-_{II} + F^-_{IA} + F^-_{IB} + F_{sI}$. Taking the difference gives $f^+(I_x) - f^-(I_x) = 0$, by the conservation condition.

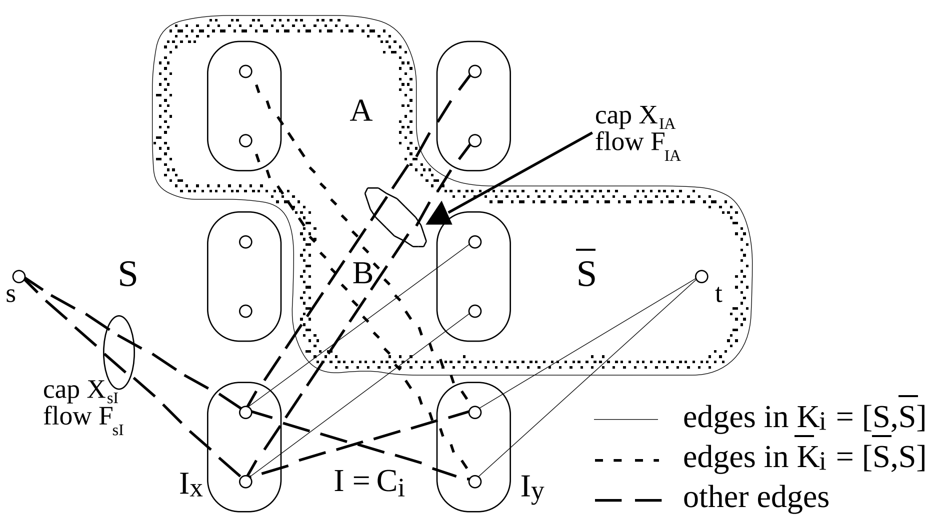

Fig. 3, The edges K_i and $\overline{K}_i$.

2.3 Lemma. Let f be a balanced flow in N, and let K be a balanced edge-cut. If $f(K_i)=0$ for some C_i, then there is an edge e K_i such that $f(e)<cap(e)$.

Proof. Referring to Fig. 3, we see that $f(\bar{K}_i)=F^-_{BI} + F^+_{AI}$ and that $f(K_i) = F^+_{IB} + F^-_{IA} + F_{It}$. By Lemma 2.2, we can add the expression $(F^+_{II} - F^-_{II}) + (F^+_{IA} - F^-_{IA}) + (F^+_{IB} - F^-_{IB}) - F_{sI}$ to $f(K_i)$ without changing its value. This gives $f(K_i) = (F^+_{II} - F^-_{II}) + F^+_{IA} + 2F^+_{IB} - F^-_{IB}$, since $F_{It}=F_{sI}$ by complementarity. If $f(\bar{K}_i)=0$ then $F^-_{BI} = F^+_{AI} = 0$, so that $f(K_i) = (F^+_{II} - F^-_{II}) + 2F^+_{IB}$. But F^+_{II} and F^-_{II} are even quantities, since e E_{II} if and only if e′ E_{II}. So $f(K_i)$ is also even. But by the definition of K, each $cap(K_i)$ is odd. Therefore $f(K_i)<cap(K_i)$. It follows that each K_i contains an edge e for which $f(e)<cap(e)$.

This shows that for each set K_i, either $f(\bar{K}_i)>0$, in which case $\bar{K}_i$ contains an edge e with $f(e)>0$, or else $f(\bar{K}_i)=0$ and K_i contains an edge e for which $f(e)<cap(e)$.

2.4 Theorem. Let $C_1, C_2, \ldots, C_k$ be the connected components of C in the balanced edge-cut $K=[S,\bar{S}]$. Then for any balanced flow f, $val(f) \le cap(K) - k$.

Proof. By Lemma 2.3, one of $cap(K_i) - f(K_i)$ and $f(\bar{K}_i)$ is at least 1, for each C_i. Therefore $(cap(K) - f(K)) + f(\bar{K}) \ge k$, which implies that $cap(K) - (f(K) - f(\bar{K})) \ge k$. But $f(K) - f(\bar{K}) = f^+(S) - f^-(S) = val(f)$. Therefore $val(f) \le cap(K) - k$.

The capacity of a balanced edge cut, denoted *balcap(K)*, is therefore defined to be $cap(K) - k$, where k is the number of connected components of N[C]. A *minimum* balanced edge-cut is a balanced edge-cut such that no balanced edge-cut has smaller capacity. We have therefore proved:

2.5 Corollary. The value of a balanced flow in a balanced network is less than or equal to the capacity of every balanced edge-cut, that is, $val(f) \le balcap(K)$, for all balanced f and K.

This holds also when f is maximum and K is minimum. In order to prove that the value of a maximum flow actually equals the capacity of a minimum balanced edge-cut, we need to consider augmenting paths.

3. The Max-Flow-Min-Cut Theorem in Balanced Networks.

In general, a flow f in a network N is maximum if and only if there is no augmenting path in N. Because of this theorem, algorithms can use augmenting paths to find a maximum flow. We need a similar theorem when N and f are balanced.

Suppose that $P=(v_1,v_2,\ldots,v_m)$ is a *path* in a balanced network N, that is, P has no repeated vertices, and consecutive vertices v_i and v_{i+1} are adjacent. Edges can appear in both directions on P, so that either $v_i \to v_{i+1}$, in which case v_iv_{i+1} is a *forward* edge of P; or else $v_{i+1} \to v_i$, and v_iv_{i+1} is a *backward* edge of P. Let e be an edge on P. The *residual capacity* of e is defined in terms of the direction in which it is traversed:

$$\text{rescap}(e) = \begin{cases} \text{cap}(e) - f(e), & \text{if } e \text{ is a forward edge,} \\ f(e), & \text{if } e \text{ is a backward edge.} \end{cases}$$

An edge e is *saturated* if rescap(e)=0. Otherwise it is unsaturated. This depends, of course, on the direction of traversal. A path P is unsaturated if *all* its edges are unsaturated. The *complementary path* of P is defined to be $P'=(v'_m, \ldots, v'_2, v'_1)$. The residual capacity of an augmenting path P is $\delta(P)=\min_{e \in P} \text{rescap}(e)$. Clearly e′ has the same residual capacity on P′ as e has on P, so δ(P′)=δ(P).

If P is a uv-path, that is, if it begins at u and ends at v, then P′ is a v′u′-path. If P is a vv′-path, *then P′ is also a vv′-path*. This is of particular use when P is an augmenting st-path, since then P′ is also an augmenting st-path. If we begin with a balanced flow f, and augment on P and on P′ as well, then every edge in N will carry the same flow as its complementary edge, so that f will still be balanced. This is illustrated in Fig. 4.

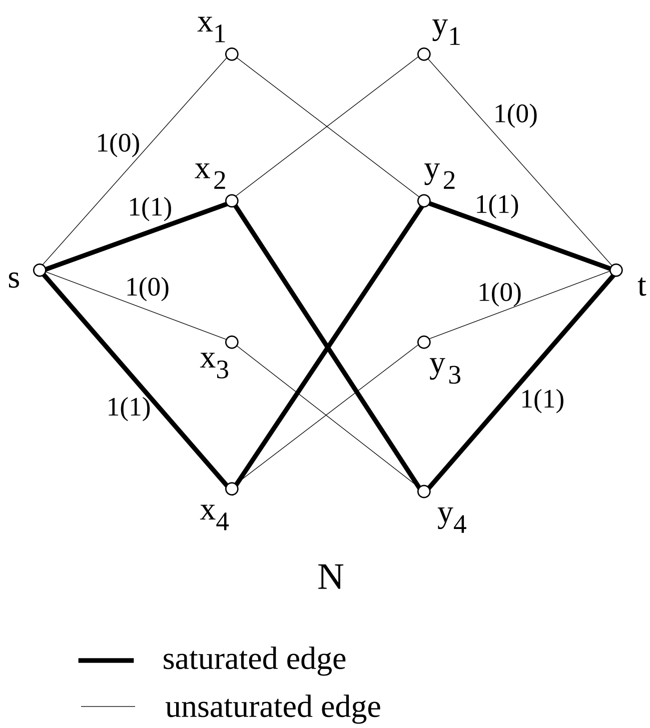

Fig. 4, Complementary augmenting paths $P=sx_1y_2x_4y_3t$ and $P'=sx_3y_4x_2y_1t$.

Suppose that P is an augmenting path. Then P′ is also an augmenting path. However we may not always be able to augment on both P and P′. It is sometimes the case that augmenting on P destroys the augmenting path P′.

3.1 Lemma. Given an augmenting path with residual capacity δ, we can augment the flow on P and P′ by at least one unit (but not necessarily by δ units), if and only if P does not contain a pair of complementary edges with a residual capacity of one.

Proof. Suppose first that P does contain such a pair of complementary edges. Then P has the form P=(s…uv…v′u′…t), where rescap(uv)=1. When we augment on P, the residual capacity of uv and v′u′ on P′ becomes zero, so that P′ is no longer an augmenting path in N. Now suppose that P does not have this form, and let e be an edge of P. If e′ is not on P, then augmenting on P does not change its flow, so its residual capacity on P′ remains ≥δ. If e′ is on P, then rescap(e′)≥2 before augmenting, so that if we augment by one unit, rescap(e′) on P′ changes by one, so we can still augment on P′.

We must therefore ensure that any algorithm we use to find a maximum flow in a balanced network N considers an unsaturated st–path to be an augmenting path only if it does not contain a pair of complementary edges with a residual capacity of one.

3.2 Definition. An unsaturated uv-path which contains a pair of complementary edges with residual capacity 1 is called an *invalid path*. Any other unsaturated uv-path is called a *valid path*, and only a valid st-path is considered to be an augmenting path in N. A vertex v is *s-reachable* if there is a valid path from s to v, and *t-reachable* if there is a valid path from v to t.

Let N and f be balanced, and suppose that there is no valid augmenting path in N. *Let S be the set of all s-reachable vertices* and consider the edge-cut $K=[S,\bar{S}]$. K has the form illustrated in Fig. 2. The vertices can be partitioned into the 4 sets A,B,C, and D. As before, let C_i denote the connected components of N[C], for i=1,2,…,k, and let K_i, K_r, $\bar{K}_i$, and $\bar{K}_r$ be as defined in section 2.

3.3 Lemma. Every edge of K_r $\bar{K}_r$ is saturated from S to $\bar{S}$.
Proof. Let (u,v) be an unsaturated edge from S to $\bar{S}$, so that u S and v $\bar{S}$. Then there is a valid su-path, but no valid sv-path. Therefore the edge (uv)′, and hence the vertex u′, must be on every valid path to u. Since u S, there is at least one such path, so both u and u′ are s-reachable; this means that u and u′ are in C, so uv is in some K_i or $\bar{K}_i$.

3.4 Lemma. There are no edges between C and D.
Proof. First notice that D $\bar{S}$, so no vertex in D is s-reachable. Suppose that such an edge $e=(x_i,y_j)$ existed, as in Fig. 5. Then $e'=(x_j,y_i)$ is also such an edge. Either these two edges carry flow or they do not. If they do, then x_jy_i is unsaturated from $\bar{S}$ to S. But then x_j would be in S, not $\bar{S}$. If they do not carry flow, then x_iy_j is unsaturated form S to $\bar{S}$, so y_j would be in S, again a contradiction.

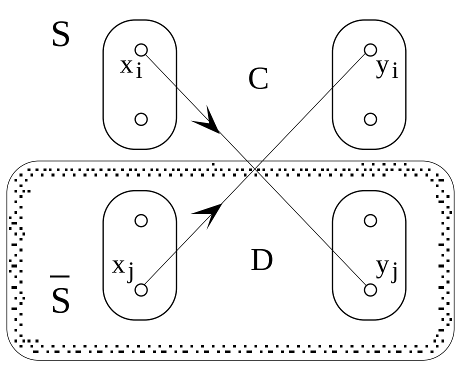

Fig. 5, Edges between C and D.

3.5 Lemma. Each C_i consists entirely of pairs of complementary vertices.

Proof. The proof is illustrated in Fig. 6. Let P be any valid path from s to w C_i , and suppose that P first enters C_i along the edge uv. Then u′ $\overline{S}$, so every valid path to v′ goes through v, and such a path exists because v′ C. But if Q is the sub–path that is a vv′-path, then Q′ is also a vv′-path. Thus every vertex on both Q and Q′ is in C, and therefore in C_i, so Q Q′ C_i. But if w C_i then there is a path W connecting either v or v′ to w, since C_i is connected, so say v. Then W′ connects w′ to v′, so w′ is also in C_i.

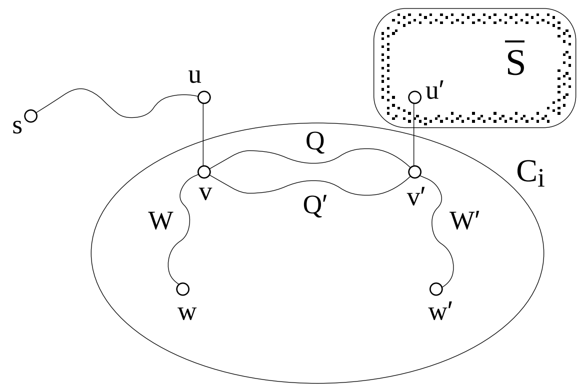

Fig. 6, Complementary vertices in C_i.

3.6 Lemma. At least one edge is unsaturated from S to $\overline{S}$ in each K_i $\overline{K}_i$, for $1 \le i \le k$.

Proof. The proof is illustrated in Figs. 6 and 7. Take an arbitrary component C_i, and any valid path P from s to a vertex in C_i. Let uv be the first edge on P where v C_i. Then (uv)′=v′u′ is unsaturated because uv is. If uv is a forward edge on P then v′u′ is an unsaturated edge from S to $\overline{S}$, and v′u′ K_i. If uv is a backward edge, then f(uv)>0, since it is unsaturated. But then v′u′ $\overline{K}_i$. In either case, K_i $\overline{K}_i$ contains an unsaturated edge.

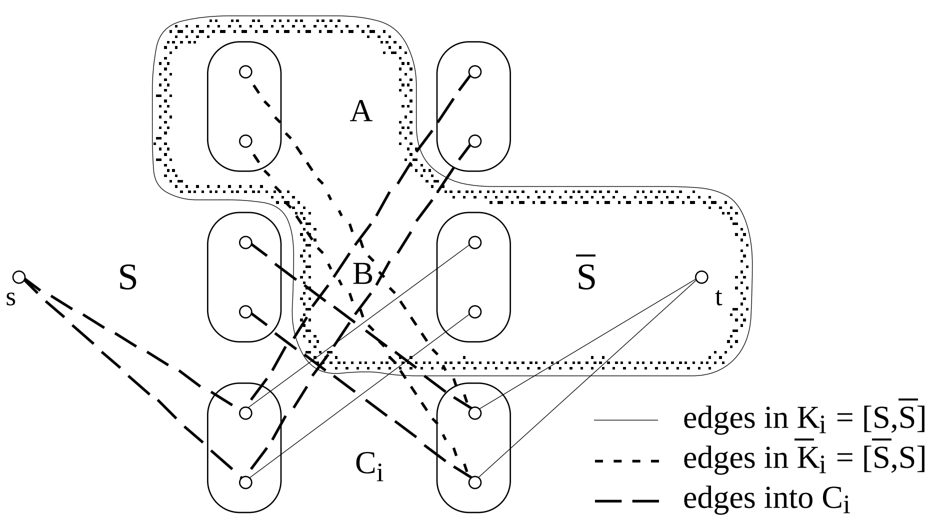

Fig. 7, Edges of K_i $\overline{K}_i$

3.7 Theorem. Let v C_i be a vertex such that there is a valid sv-path which first enters C_i at v. Then every w C_i can be reached on a valid path Q_w from v, where Q_w C_i.

Proof. By contradiction. Refer to Figs. 8 and 9. Let C^* be the largest non empty set of pairs of complementary vertices in C_i, with the property that every vertex w C^* is reachable from v on a valid path. The paths Q and Q′ of Lemma 3.5 are in C^*, so $C^*\neq\emptyset$.

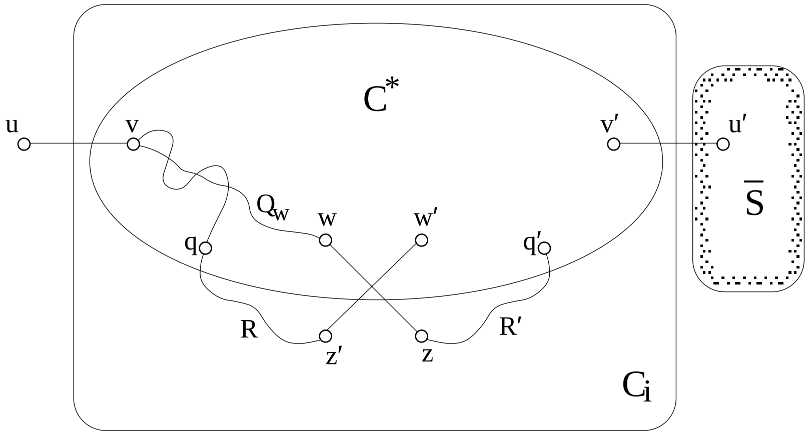

Fig. 8, C^* and C_i, v P.

Now suppose that $C^*\neq C_i$, so there is some edge wz such that w C^* and z C_i–C^*; thus w′ C^* and z′ C_i–C^*. If wz were saturated, then zw, and hence w′z′, would be unsaturated. Thus either wz or w′z′ is an unsaturated edge from C^* to C_i–C^*, so suppose it is wz. z′ C_i, so there is a valid sz′-path P.

Suppose first that v P. v C^* and z′ C^*, so let P leave C^* for the last time from a vertex q, and let R be the portion of P from q to z′. If wz is on R, then q=w, z R, and the subsequent portion of R is a zz′-path. Then its complement would also be a zz′-path. But then we can then enlarge C^* with the vertices of R R′, a contradiction. Otherwise wz is not on R. Let Q_w be a valid path from v to w, contained in C^*. R′ is a valid path because R is. The concatenation $Q_w$wzR′ is a valid path because if it were not, then R′ would contain an edge e such that either e′=wz or e′ is on Q_w. But R contains no edge of Q_w , since Q_w C^*, and wz is not on R. But then $Q_w$wzR′ is a valid path and we can again enlarge C^* with the vertices of R R′, a contradiction.

Otherwise v P. Then Pz′w′ is a valid path into C^*. Let q′ be the first vertex at which P enters C^* (maybe q′=w′), and let $P_{q'}$ be the portion of P up to q′. Then $P_{q'}Q'_q$ is a valid sv′-path that does not contain v. This is a contradiction, so $C^*=C_i$, as required.

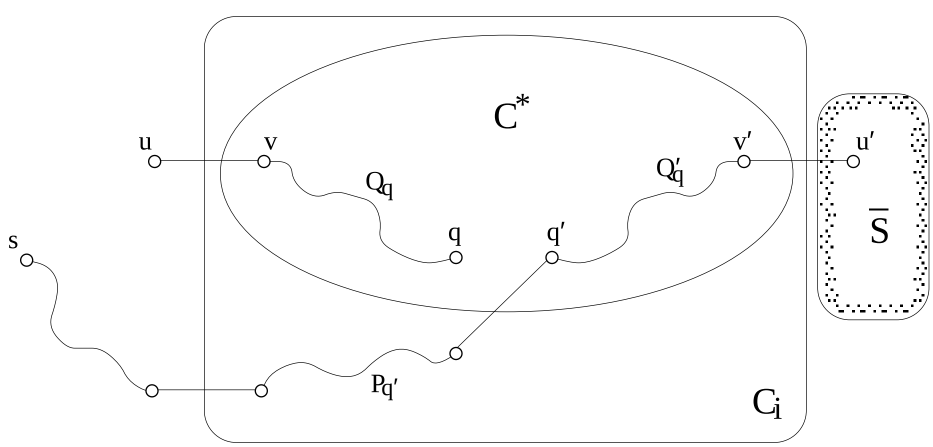

Fig. 9, C^* and C_i, v P.

3.8 Lemma. At most one edge is unsaturated, with residual capacity at most 1, from S to $\bar{S}$, in each K_i $\bar{K}_i$, for $1 \le i \le k$.

Proof. If any edge e of K_i $\bar{K}_i$ were unsaturated from S to $\bar{S}$, with rescap(e)>1, then a vertex in $\bar{S}$ would be s-reachable, a contradiction. By lemma 3.6 there is always at least one unsaturated edge v′u′, so suppose there is a second, say z′w′, as shown in Fig. 10. Then if R is a valid sv-path and Q is valid vz′-path contained in C_i, which both exist, then RQ would be a valid path to z′ that does not use the edge wz, a contradiction.

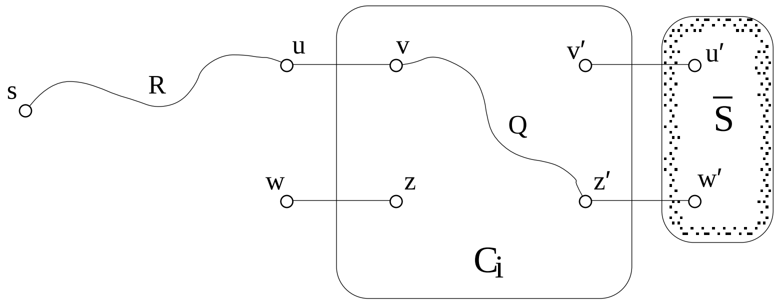

Fig. 10, A component C_i.

This proves that there is exactly one edge in each K_i $\bar{K}_i$, for $1 \le i \le k$, that is unsaturated from S to $\bar{S}$. But by lemma 3.3, every edge in K_r $\bar{K}_r$ is saturated from S to $\bar{S}$, so we have proved:

3.9 Corollary. Let $K=[S,\bar{S}]$, where S is the set of all s-reachable vertices, as above. Then

$$\mathrm{val}(f) = \mathrm{cap}(K) - k.$$

3.10 Lemma. $\mathrm{cap}(K_i)$ is odd, for $1 \le i \le k$.

Proof. By lemmas 3.6 and 3.8, either $f(K_i)=\mathrm{cap}(K_i)$ and $f(\bar{K}_i)=1$, or $f(K_i)=\mathrm{cap}(K_i)-1$ and $f(\bar{K}_i)=0$. Referring to the proof of lemma 2.3, we have $f(\bar{K}_i)=F^-_{BI} + F^+_{AI}$ and $f(K_i) = (F^+_{I1} - F^-_{I1}) + F^+_{IA} + 2F^+_{IB} - F^-_{IB}$. In the first case, $f(\bar{K}_i)=F^-_{BI} + F^+_{AI}=1$, so one of F^+_{AI} and F^-_{BI} is 0 and the other is 1. But then $f(K_i) = \mathrm{cap}(K_i) = (F^+_{I1} - F^-_{I1}) + F^+_{IA} + 2F^+_{IB} - F^-_{IB}$, which is an odd number, since F^+_{I1} and F^-_{I1} are even. In the second case, $f(\bar{K}_i)=F^-_{BI} + F^+_{AI}=0$, so that $f(K_i) = \mathrm{cap}(K_i)-1 = (F^+_{I1} - F^-_{I1}) + 2F^+_{IB}$ is even. It follows that $\mathrm{cap}(K_i)$ is odd.

3.11 Corollary. K is a balanced edge-cut.

Proof. There are no edges between C_i and C_j , for any i and j, since the C_i are connected components, so by lemmas 3.4, 3.5, and 3.10, K is a balanced edge-cut, and balcap(K)=cap(K)–k.

But val(f)=cap(K)–k, by corollary 3.9, so by theorem 2.4, f is a maximum balanced flow in N, and K is a minimum balanced edge-cut. We therefore have:

3.12 Theorem. Let N be a balanced network with a balanced flow f. Then f is maximum if and only if there is no valid augmenting path in N.

Proof. If there is a valid augmenting path, then by lemma 3.1, f is not maximum. If there is no valid augmenting path, then by corollary 3.11, the set S of s-reachable vertices defines a balanced edge-cut $K=[S,\overline{S}]$ for which val(f)=cap(K)–k, so that f is maximum, by theorem 2.4. The above results can be summarized in a theorem.

Max-Balanced-Flow-Min-Balanced-Cut Theorem. Let N be a balanced network. The value of a maximum balanced flow equals the capacity of a minimum balanced edge-cut.
Proof. By theorem 2.4 and corollaries 3.9 and 3.11.

These results mean that in a balanced network N, we can begin with the zero flow and successively augment the flow on complementary augmenting paths until no valid augmenting path remains. At that point the flow will be maximum, and the set of s-reachable vertices will define a balanced edge-cut. Such an algorithm is developed in Kocay and Stone [4]. One result is that *maximum matchings, f-factors, and capacitated b-matchings can all be found by a single flow algorithm*. When the flow is maximum, a balanced edge-cut will be given by the set of s-reachable vertices.

4. The Factor Problem.

Given the function b(v) of section 1, a subgraph H such that $\deg_H(v)=b(v)$ is called a *factor* of G. (Usually H is called an f-factor, since the function b(v) is usually denoted f(v), but we are using f to denote a flow.) Given arbitrary sets $A,B \subseteq V(G)$, let $|A,B|$ stand for the number of edges with one end in A and one in B. Let $S,T \subseteq V(G)$. Write odd(S,T) for the number of components C of $G-(S \cup T)$ such that $|T,C| + \sum_{v \in C} b(v)$ is odd. Tutte's Factor Theorem [10] can be stated as follows.

4.1 Factor Theorem. G is without a factor if and only if there is a subset S of V(G) and a subset T of V–S such that

$$\sum_{v \in S} b(v) < \mathrm{odd}(S,T) + \sum_{v \in T} (b(v) - \deg_{G-S}(v)).$$

We show how this can be derived from the Max-Balanced-Flow-Min-Balanced-Cut Theorem. Let $V(G)=\{v_1, v_2, \ldots, v_n\}$. Write $2\varepsilon = \sum_{v \in V(G)} b(v)$. Create a balanced network N with vertices $X=\{x_1, x_2, \ldots, x_n\}$ and $Y=\{x_1, x_2, \ldots, x_n\}$, as well as s and t. N has edges (x_i,y_j) and (x_j,y_i) for every edge v_iv_j of G, and all edges (s,x_i) and (y_i,t) for all vertices v_i of G. So G is a homomorphic image of the [X,Y]-edges of N. The capacities are $cap(sx_i)=cap(y_it)=b(v_i)$, and $cap(x_iy_j)=cap(x_jy_i)=1$. Given any balanced flow f in N, the flow-carrying edges of [X,Y] define a subgraph H of G such that $\deg_H(v_i)=f(sx_i)$.

4.2 Lemma. G has a factor H if and only if balcap(K) ≥ 2ε, for all balanced edge-cuts K.
Proof. The number of edges of H is twice val(f).

Let f be a maximum balanced flow, and let K be the corresponding balanced edge-cut which defines the sets A,B,C, and D in N. Let A^*, B^*, C^*, D^* be the corresponding sets of V(G). C_1, C_2, …, C_k denote the connected components of N[C], and each K_i has odd capacity, by 3.10. We use the shorthand notation $\sum A = \sum_{v \in A^*} b(v)$, and so forth, for the sets B, C, D.

4.3 Lemma. $\mathrm{balcap}(K) = \sum A - \sum B + 2\varepsilon + \sum_{v \in B^*} \deg_{G-A^*}(v) - k$.

Proof. Referring to Fig. 2, $\mathrm{balcap}(K) = 2\sum A + |B,B| + |B,C| + |B,D| + \sum C + \sum D - k$. Notice that $2\varepsilon = \sum A + \sum B + \sum C + \sum D$ and that $|B,B|+|B,C|+|B,D| = \sum_{v \in B^*} \deg_{G-A^*}(v)$. The result follows.

4.4 Proof of the Factor Theorem. Suppose that G is without a factor. Then there is a balanced edge-cut such that $\mathrm{balcap}(K)<2\varepsilon$. By lemma 4.3, this gives $\sum A - \sum B + \sum_{v \in B^*} \deg_{G-A^*}(v) - k < 0$. Each K_i has odd capacity. Notice that $\mathrm{cap}(K_i) = |B,C_i| + \sum_{v \in C_i^*} b(v)$. Therefore, $k \le \mathrm{odd}(A^*,B^*)$. It follows that $\sum_{v \in A^*} b(v) < \mathrm{odd}(A^*,B^*) + \sum_{v \in B^*} (b(v) - \deg_{G-A^*}(v))$.

Conversely, suppose that $\sum_{v \in A^*} b(v) < \mathrm{odd}(A^*,B^*) + \sum_{v \in B^*} (b(v) - \deg_{G-A^*}(v))$ for disjoint sets A^* and B^* of V(G). Form a balanced edge-cut by taking C^* to consist of all components of $G-(A^* \cup B^*)$ such that $|B,C| + \sum_{v \in C^*} b(v)$ is odd. D^* contains the remaining vertices of G. It is easy to verify that this determines a balanced edge-cut K of N for which $\mathrm{balcap}(K)<2\varepsilon$.

Thus we have proved the Factor Theorem with balanced networks. In the case that no factor exists in G, the maximum balanced flow algorithm actually *finds* the sets A^* and B^*; they are part of the edge-cut formed by the set of all s-reachable vertices. That is, it either finds the factor or finds the two sets that prove that no factor exists.

Suppose that edges of G may be used more than once by edges of H, so that H is a multigraph, with maximum multiplicity m allowed. Then $\mathrm{balcap}(K) = \sum A - \sum B + 2\varepsilon + m\sum_{v \in B^*} \deg_{G-A^*}(v) - k$, so that G is without a factor H if and only if there are sets A^* and B^* such that $\sum_{v \in A^*} b(v) < \mathrm{odd}_m(A^*,B^*) + \sum_{v \in B^*} (b(v) - m \deg_{G-A^*}(v))$,
where $\mathrm{odd}_m(A^*,B^*)$ is the number of components C^* of $G-(A^* \cup B^*)$ such that $m|B,C^*| + \sum_{v \in C^*} b(v)$ is odd.

When m is allowed to approach ∞, so that edges may be used any number of times in H, we obtain Tutte's Solubility Theorem [10]. In this case, [B,B]=[B,C]=[B,D]=Ø, for otherwise balcap(K) would be infinite. So G is not soluble if and only if there exist disjoint sets A^*,B^* V(G) such that $\sum A < \sum B + k$, where k is the number of components C^* of G–(A^* B^*) such that $\sum_{v\ C^*} b(v)$ is odd.

The Erdös-Gallai conditions [2] for the existence of a simple graph with prescribed degree sequence and prescribed maximum edge-multiplicity can be derived directly from the max-balanced-flow-min-balanced-cut theorem. Similarly it can be used to find a formula for the number of edges in a maximum matching in an arbitrary graph.

In summary, Tutte's Factor Theorem can be viewed as a min-max theorem of flow theory. The theory of f-barriers [6,9,10] can be viewed as the theory of edge-cuts in balanced networks. As a result, the theoretical and algorithmic techniques of flow theory can be used for solving these kinds of problems. Balanced networks provide a simplification of the theory of f-factors, f-barriers, and capacitated b-matchings by placing them into the context of network flows.

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