On 3-Hypergraphs with Equal Degree Sequences

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Abstract

The question of necessary and sufficient conditions for the existence of a simple 3-uniform hypergraph with a given degree sequence is a long outstanding open question. We provide a result on degree sequences of 3-hypergraphs which shows that any two 3-hypergraphs with the same degree sequence can be transformed into each other using a sequence of trades, also known as null-3-hypergraphs. This result is similar to the Havel-Hakimi theorem for degree sequences of graphs.

1 Introduction

The question of necessary and sufficient conditions for the existence of a simple hypergraph with a given degree sequence is a long outstanding open question. See Berge [1], and Murthy and Srinivasan [2]. In Colbourn, Kocay and Stinson [4], it was proved that certain related questions are NP-complete. Many graph problems that have

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polynomial-time algorithms are known to be NP-complete when applied to 3-uniform hypergraphs. One example is the determination of the existence of a perfect matching in a graph (which is in P) versus the existence of a 3D-matching in a 3-uniform hypergraph (which is NP-complete). However, given a sequence of n positive integers, the computational complexity of determining whether there is a simple 3-uniform hypergraph with this sequence as its degree sequence, is currently unknown. In this paper, we present a result which may be useful in resolving this problem.

The set of all k-subsets of a set V is denoted by $\binom{V}{k}$. A *simple* k-uniform hypergraph on the vertex set V is any subset $H \subseteq \binom{V}{k}$ (repeated k-sets are not allowed). $\binom{V}{k}$ is also called the *complete* k-uniform hypergraph, as it contains all k-sets. In this paper we are concerned with $k = 3$. By the term 3-*hypergraph*, we will always mean a simple 3-uniform hypergraph. Each 3-set $X \in \binom{V}{3}$ is called a *triple*. We will also use the term *triple system* for 3-hypergraph. Given any $x \in V$, the degree of x in a hypergraph H is $\deg(x, H)$, the number of triples of H which contain x. Let $V = \{1, 2, \ldots, n\}$. The *degree sequence* of H is $(d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \ldots \geq d_n$ are the degrees of the vertices. A sequence $D = (d_1, d_2, \ldots, d_n)$ of integers, such that $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$ is called *valid* or *hypergraphic* if there is a simple 3-hypergraph H with degree sequence D.

3-Hypergraph Degree Sequence: Given a sequence $D = (d_1, d_2, \ldots, d_n)$ of integers, such that $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$.
Question: Is D a hypergraphic degree sequence?

This question of whether there exists a polynomial-time algorithm to settle this question has remained unsolved for many years. It is stated as Problem 3.1 in [4]. The corresponding question for graphs was solved by the Havel-Hakimi theorem [7] and the Erdös-Gallai conditions [7]. The Havel-Hakimi theorem results in a polynomial-time algorithm to construct a simple graph with a given degree sequence, if one exists. The Erdös-Gallai conditions characterize the polytope of valid degree sequences. A polynomial-time algorithm to solve the hypergraph degree sequence problem, even when restricted to 3-uniform hypergraphs, is unknown.

In Section 2, we discuss trades and state our main result. In Section 3 we present the proof of our main result. In Section 4, we discuss the problem of partitioning 3-hypergraphs into two 1-designs.

2 Trades, Null-Hypergraphs

Let H_1 and H_2 be two 3-hypergraphs on the set V such that $\deg(x, H_1) = \deg(x, H_2)$, for all $x \in V$. We assign a weight of $+1$ to each triple in H_1, and -1 to each triple in H_2. Let $H = H_1 \oplus H_2$ be the *exclusive or* (also known as *symmetric difference*) of the sets of triples of H_1 and H_2. Thus, a triple belongs to H if and only if it belongs to exactly one of H_1 or H_2. It may be that $H = \emptyset$, in which case $H_1 = H_2$. Otherwise, H consists of a number of triples having a weight of $+1$ and an equal number of triples having a weight -1. Then the net degree of any vertex x, taking the weights into consideration, is $\deg(x, H) = \deg(x, H_1) - \deg(x, H_2) = 0$. Any weighted hypergraph H whose triples have been assigned weights ± 1, with the property that $\deg(x, H) = 0$ for all $x \in V$ is called a *null hypergraph*. We will also use the terms *null triple system* and *trade*, although null hypergraph is more general. The term *trade* derives from design theory – when the triples of H_2 are removed and substituted with the triples of H_1, a "trade" has occurred, but the degrees of the vertices have not changed. The book *Triple Sytems* by Colbourn and Rosa [3] contains a section on trades in Steiner triple systems. We will use the word "trade" in this sense, when a set of triples is removed from a hypergraph, and substituted with another set, so as to maintain the vertex degrees. We now proceed to look at null 3-hypergraphs on small vertex sets.

It is fairly easy to see that there are no null triple systems when $|V| \leq 4$ (except the *empty* hypergraph, containing no triples). There are three possible null triple systems when $|V| = 5$. They are shown in Figure 1, where a triple $\{i, j, k\}$ is denoted ijk. The fact that these are the only null triple systems on 5 vertices (up to isomorphism) was verified by an exhaustive computer search. We denote the first of these three null triple systems by N_5. When we need to indicate the triples it contains, we also write it as $N_5(123, 145; 125, 134)$, where the first set of triples are those with positive weight, and the second set are those with negative weight.

On 6 vertices, there are many null triple systems (a non-exhaustive

$H_1:$	$123, 145$	$H_1:$	$123, 245, 345$	$H_1:$	$123, 135, 145, 234, 245$
$H_2:$	$125, 134$	$H_2:$	$145, 234, 235$	$H_2:$	$134, 124, 125, 235, 345$

Figure 1: N_5, N_a and N_b, the null triple systems on five vertices

search has found 83). We will only be concerned with a single null triple system on 6 vertices, given by Figure 2.

$H_1:$	$123, 456$
$H_2:$	$124, 356$

Figure 2: N_6, a null triple system on six vertices

We denote this null triple system by N_6. When we need to refer to the actual triples, we write it as $N_6(123, 456; 124, 356)$. These null triple systems can also be represented by bipartite incidence graphs, as shown in Figure 3, where the nodes coloured black represent the triples, and the nodes containing numbers represent the vertices.

Notice that if N and N' are null triple systems on vertices V, such that all triples of $N \cap N'$ have opposite sign in N and N', then $N \oplus N'$ is also a null triple system. We now state the main theoretical result of this paper.

Theorem 2.1 *Let H be any null 3-hypergraph on vertex set V. Then there is a sequence of null 3-hypergraphs $M_1, M_2, \ldots, M_k$, for some $k \geq 0$, such that $H = M_1 \oplus M_2 \oplus \ldots \oplus M_k$, where each M_i is isomorphic to either N_5 or N_6.*

It will follow from this theorem, that if H_1 and H_2 are any two 3-hypergraphs with the same vertex degrees, and a null triple system $H_1 \oplus H_2$ is created by assigning weight $+1$ to the triples of H_1; and -1 to the triples of H_2, so that $H_1 \oplus H_2 = M_1 \oplus M_2 \oplus \ldots \oplus M_k$; then $H_2 = H_1 \oplus M_1 \oplus M_2 \oplus \ldots \oplus M_k$; that is, any 3-hypergraph H_2 with the same vertex degrees as H_1 can be constructed from H_1 by a sequence of trades isomorphic to N_5 or N_6. At each step in the transformation, two triples are removed, and two are added, such

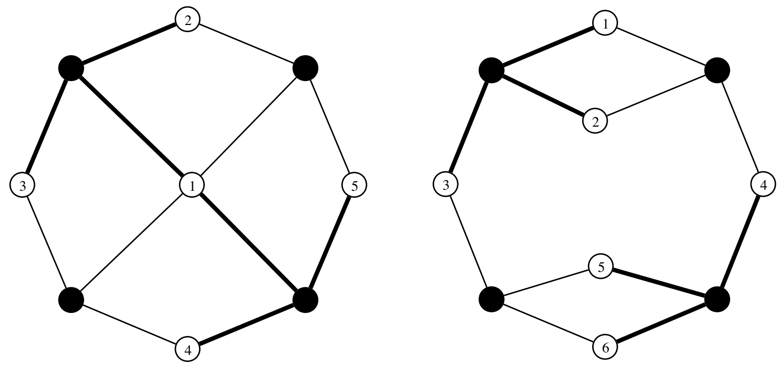

Figure 3: The incidence graphs of N_5 and N_6

that they form an N_5 or N_6. Moreover, the proof will show how to transform H_1 into H_2.

3 Proof of the Theorem

Let H be a null 3-hypergraph on vertex set V with n vertices. Let b denote the number of triples of H with positive weight. The proof of Theorem 2.1 is by induction on b. If $b = 0$, the result is clearly true. As there are no null triple systems with $b = 1$, it is also true when $b = 1$. The only null triple systems with $b = 2$ are N_5 and N_6. Therefore, it is also true when $b = 2$. Suppose now that $b \geq 3$. We show that H can always be reduced to a null triple system H' with $b' < b$ positive triples.

It is convenient to colour the edges of H with weight $+1$ blue, and those with weight -1 red. We then denote a blue triple as $B123$, etc. We write the blue degree of a vertex x as $\deg_B(x, H)$, and the red degree as $\deg_R(x, H)$.

Case 1. There is a red triple and a blue triple intersecting in two vertices.

Without loss of generality, let the triples be $B123$ and $R124$. There must also be a red triple containing vertex 3, and a blue triple containing vertex 4.

(1a) There exists a red triple T containing vertex 3, but not vertex 4.

Let $T = R3uv$, where $u, v \neq 4$. Notice that $\{u, v\} \neq \{1, 2\}$, since $B123$ is a blue triple. Therefore $\{u, v\}$ and $\{1, 2\}$ intersect in at most one vertex.

(1a.i) There is no red triple $R4uv$.
Let $N = N(124, 3uv; 123, 4uv)$. If $\{u, v\} \cap \{1, 2\} = \emptyset$, this will be an N_6. Otherwise it will be an N_5. Set $H' = H \oplus N$, as shown in Figure 4. Then H' is a null triple system with at most $b - 2$ blue triples. By induction, H' can be written as $M_1 \oplus M_2 \oplus M_3 ... \oplus M_k$, so that $H = N \oplus H'$, as required.

$H :\quad B123; R124, R3uv, \ldots$
$N :\quad B124, B3uv; R123, R4uv$
$H \oplus N :\quad R4uv, \ldots$

Figure 4: Case 1a.i

(1a.ii) Otherwise, for every red triple $R3uv$, where $u, v \neq 4$, there is also a red triple $R4uv$. Since $R124$ is a red triple, it follows that $\deg_R(4, H) \geq \deg_R(3, H) + 1$. We will return to this case shortly.

(1b) There exists a blue triple T containing vertex 4, but not vertex 3.

This case is symmetric to (1a). However, we will later need the explicit statement of the conclusion of (1b.ii); hence we include the details of the proof. Let $T = B4uv$, where $u, v \neq 3$. Notice that $\{u, v\} \neq \{1, 2\}$, since $R124$ is a red triple. Therefore $\{u, v\}$ and $\{1, 2\}$ intersect in at most one vertex.

(1b.i) There is no blue triple $B3uv$.
Let $N = N(124, 3uv; 123, 4uv)$. If $\{u, v\} \cap \{1, 2\} = \emptyset$, this will be an N_6. Otherwise it will be an N_5. Set $H' = H \oplus N$, as shown in Figure 5. Then H' is a null triple system with at most $b - 2$ blue triples. By

induction, H' can be written as $M_1 \oplus M_2 \oplus M_3 ... \oplus M_k$, so that $H = N \oplus H'$, as required.

$$\begin{array}{ll} H: & B123, B4uv; R124, \ldots \\ N: & B124, B3uv; R123, R4uv \\ H \oplus N: & B3uv, \ldots \end{array}$$

Figure 5: Case 1b.i

(1b.ii) Otherwise, for every blue triple $B4uv$, where $u, v \neq 3$, there is also a blue triple $B3uv$. Since $B123$ is a blue triple, it follows that $\deg_B(3, H) \geq \deg_B(4, H) + 1$. We will also return to this case shortly.

(1c) Every red triple containing vertex 3 also contains vertex 4 and every blue triple containing vertex 4 also contains vertex 3.

In this case, since $R124$ is a red triple, we have $\deg_R(4, H) \geq \deg_R(3, H) + 1$. Also, since $B123$ is a blue triple, we have $\deg_B(3, H) \geq \deg_B(4, H) + 1$. Since $\deg_R(4, H) = \deg_B(4, H)$ and $\deg_R(3, H) = \deg_B(3, H)$, this case is clearly impossible.

To complete the proof of Case 1, we notice that if one of (1a.i) or (1b.i) occurs, that there is a trade N_5 or N_6 which reduces H to H' with fewer triples, so that induction can be used. In all other cases (1a.ii), (1b.ii), we have $\deg_R(4) \geq \deg_R(3, H) + 1$ and $\deg_B(3, H) \geq \deg_B(4, H) + 1$. But since $\deg_R(3, H) = \deg_B(3, H)$ and $\deg_R(4, H) = \deg_B(4, H)$, this is impossible. Therefore, we see that cases (1a.ii) and (1b.ii) are also impossible. We conclude that at least one of cases (1a.i) or (1b.i) always applies.

Case 2. Any red triple and blue triple intersect in at most one vertex.

Notice that as H is a null 3-hypergraph, there must exist two triples (of different colors) that intersect in exactly one vertex. Without loss of generality, take a blue triple $B123$ and a red triple $R145$, which intersect in vertex 1. There is also a red triple containing vertex 2 and a blue triple containing vertex 4.

(2a) There exists a red triple $T = R2uv$ such that $\{u, v\} \neq \{4, 5\}$.

Without loss of generality, write $T = R26u$.

(2a.i) There is no red triple $R456$.

Let $N = N_6(145, 236; 123, 456)$. Notice that $B236 \notin H$, since $B236$ and $R26u$ intersect in two vertices. Set $H' = H \oplus N$, as shown in Figure 6. Then H' is a null 3-hypergraph on b triples, containing the triples $R26u$ and $B236$, which intersect in two vertices. By Case (1), H' can be written as $M_1 \oplus M_2 \oplus M_3 ... \oplus M_k$, so that $H = N \oplus H'$, as required.

$H:\quad B123; R145, R26u, \ldots$
$N:\quad B145, B236; R123, R456$
$H \oplus N:\quad B236; R456, R26u \ldots$

Figure 6: Case 2a.i

(2a.ii) Otherwise, for every red triple $R2uv$, where $\{u, v\} \neq \{4, 5\}$, there is also a red triple $R45u$. Since $R145$ is a red triple, it follows that $\deg_R(4, H) \geq \deg_R(2, H) + 1$. We will return to this case shortly.

(2b) There exists a blue triple $T = B4uv$ such that $\{u, v\} \neq \{2, 3\}$.

This case is symmetric to (2a). As we will later need the explicit statement of (2b.ii), we include the details of the proof. Without loss of generality, write $T = B46u$.

(2b.i) There is no blue triple $B236$.

Let $N = N_6(145, 236; 123, 456)$. Notice that $R456 \notin H$, since $R456$ and $B46u$ intersect in two vertices. Set $H' = H \oplus N$. Then H' is a null triple system on b triples, containing the triples $R456$ and $B46u$, which intersect in two vertices. By Case (1), H' can be written as $M_1 \oplus M_2 \oplus M_3 ... \oplus M_k$, so that $H = N \oplus H'$, as required.

(2b.ii) Otherwise, for every blue triple $B4uv$, where $\{u, v\} \neq \{2, 3\}$, there is also a blue triple $B23u$. Since $B123$ is a

$H:\quad B123, B46u; R145, \ldots$
$N:\quad B145, B236; R123, R456$
$H \oplus N:\quad B236, B46u; R456 \ldots$

Figure 7: Case 2b.i

blue triple, it follows that $\deg_B(2, H) \geq \deg_B(4, H) + 1$. We will also return to this case shortly.

(2c) $R245$ is the only red triple containing 2.
$B234$ is the only blue triple containing 4.
Since $R145$ is a red triple, we have $\deg_R(4, H) \geq \deg_R(2, H) + 1$. Since $B123$ is a blue triple, we have $\deg_B(2, H) \geq \deg_B(4, H) + 1$.

To complete the proof of Case 2, we notice that if one of (2a.i) or (2b.i) occurs, that there is a trade N_6 which reduces H to H' containing a red triple and a blue triple which intersect in two vertices, so that Case (1) can be used. In all other cases, we have $\deg_R(4, H) \geq \deg_R(2, H) + 1$ and $\deg_B(2, H) \geq \deg_B(4, H) + 1$. But since $\deg_R(2, H) = \deg_B(2, H)$ and $\deg_R(4, H) = \deg_B(4, H)$, this is impossible. We conclude that at least one of cases (2a.i) or (2b.i) always applies. It follows that the result is true if any red triple and blue triple intersect in at most one vertex.

This completes the proof of the theorem. □

We state the corollary as follows:

Corollary 3.1 *Let H_1 and H_2 be any two simple 3-hypergraphs with the same degree sequence. Then H_1 can be transformed into H_2 by a sequence of trades isomorphic to N_5 or N_6.*

We remark that the corresponding results for simple graphs use the single null graph $N_4(12, 34; 13, 24)$. Any two graphs with the same degree sequence can be transformed into each other using trades isomorphic to N_4. This is a consequence of the Havel-Hakimi theorem [7]. A null graph can be viewed as an Eulerian graph in which half the edges at each vertex have been coloured blue, and half have been coloured red. Such a graph always has an alternating Euler tour; that is, an Euler tour whose edges alternate blue and red. Conversely, any Euler tour in an Eulerian graph with an even number of edges can be coloured alternately red and blue to construct a null graph. This is a structural characterization of null graphs. We know of no such characterization for null 3-hypergraphs.

Problem 4.2. Find a characterization of null 3-hypergraphs.

Consider the following related problem for hypergraph degee sequences. We are given non-negative integers d_{ij}, where $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, and $d_{ij} = d_{ji}$. We ask whether there is a simple 3-hypergraph H such that i and j occur together in exactly d_{ij} triples. This problem was proved to be NP-complete in [4] (where it appears as Problem 3.3).

It is likely that a result similar to Theorem 2.1 holds for k-hypergraphs, for all $k \geq 4$. We have used the null hypergraphs N_5 and N_6 to transform a 3-hypergraph with a given degree sequence into any other with the same degree sequence. Now N_5 and N_6 are the unique null triple systems with $b = 2$ positive triples. We conjecture that the trades required for transforming k-hypergraphs with the same degree sequence, are exactly those null k-hypergraphs with $b = 2$ positive k-sets.

4 Partitioning 3-hypergraphs into 1-designs

A graph is said to be *r-regular* if every vertex has degree r. A k-uniform hypergraph in which every vertex has the same degree is said to be a *1-design*. We write 1-(v, k, λ) to denote a simple k-uniform hypergraph which is a 1-design on v vertices in which each vertex has degree λ. The three non-isomorphic null triple systems on five vertices are shown in Figure 1. The first of these is N_5. Denote the other two by N_a and N_b, respectively. Each has a set of triples H_1 of positive weight and a set H_2 of negative weight. Denote the triples of positive weight by N_a^+, and those of negative weight by

N_a^-. Notice that N_b^+ and N_b^- are both 1-$(5,3,3)$ designs with vertex set $V = \{1,2,3,4,5\}$.

Now in the complete hypergraph $\binom{V}{3}$, every vertex has degree $\binom{4}{2} = 6$. Thus, N_b^+ and N_b^- are both exactly half of $\binom{V}{3}$. In general, let $V = \{1,2,\ldots,v\}$. Whenever there is a 1-$(v,3,\frac{1}{2}\binom{v-1}{2})$ design H_1, there will be a null 3-hypergraph whose positive triples are H_1, and whose negative triples are $H_2 = \binom{V}{3} - H_1$, such that H_1 and H_2 are both 1-designs; that is, the complete hypergraph $\binom{V}{3}$ can be decomposed into two 1-designs.

Referring to Figure 1, notice that $N_5^+ \subseteq N_b^+$ and that $N_5^- \subseteq N_b^-$. It follows that $N_b - N_5$ is also a null triple system. In fact, it is isomorphic to N_a. Hence, we can view N_5 and N_a as complementary null 3-hypergraphs, with respect to the decomposition of $\binom{V}{3}$ into two 1-designs. Whenever the complete hypergraph $\binom{V}{3}$ can be partitioned into two 1-designs, there will be a relation of complementarity for null 3-hypergraphs with vertex set V. The following conjecture seems plausible.

Conjecture 4.1 *Suppose that $\binom{V}{3}$ can be partitioned into two 1-designs. Let N be any null 3-hypergraph with vertex set V. Then there is a 1-$(v,3,\frac{1}{2}\binom{v-1}{2})$ design H_1 such that $N^+ \subseteq H_1$ and $N^- \subseteq \binom{V}{3} - H_1$ (ie, N has a complement with respect to the null 3-hypergraph determined by H_1 and its complement).*

Suppose that H is a 1-$(v,3,\frac{1}{2}\binom{v-1}{2})$ design. Since $\frac{1}{2}\binom{v-1}{2}$ must be an integer, we have $(v-1)(v-2) \equiv 0 \pmod 4$, so that $v \equiv 1$ or $2 \pmod 4$. The number of triples in H is $\frac{1}{2}\binom{v}{3}$, which is then always integral, as one of $v, v-1, v-2$ is always divisible by 3. We show that this is the only requirement for the existence of a 1-$(v,3,\frac{1}{2}\binom{v-1}{2})$ design.

A 1-*factor* of a 3-hypergraph H with v vertices is any sub-hypergraph that is a 1-$(v,3,1)$ design. A 1-*factorization* is a partition of H into 1-factors. A consequence of Baranyai's theorem (see [6]) is that when $v \equiv 0 \pmod 3$, then $\binom{V}{3}$ has a 1-factorization. If in addition, $v \equiv 1$ or $2 \pmod 4$, we then choose any $\frac{1}{2}\binom{v-1}{2}$ 1-factors of a 1-factorization to obtain the required 1-design.

When $v \not\equiv 0 \pmod 3$, we can proceed as follows. Take $V = \{1,2,\ldots,v\}$. One of $v-1$ and $v-2$ is divisible by 3, so that $\frac{1}{2}\binom{v-1}{2}$

is also divisible by 3. Let $\frac{1}{2}\binom{v-1}{2} = 3m$. Choose a triple $\{a, b, c\}$, and construct the related triples $\{a+1, b+1, c+1\}, \{a+2, b+2, c+2\}, \ldots \{a+v-1, b+v-1, c+v-1\}$, where addition is reduced, modulo v, to give a unique integer in V. These triples form a 1-$(v, 3, 3)$ design. All triples whose three differences are the values $a-b, a-c, b-c$ are included in this 1-design. Now choose any other triple $\{a', b', c'\}$ such that $\{a-b, a-c, b-c\} \neq \{a'-b', a'-c', b'-c'\}$ and repeat. Do this m times until we have a 1-$(v, 3, 3m)$ design, as required. We summarize this as follows:

Theorem 4.2 $\binom{V}{3}$ *can be partitioned into two 1-*$(v, 3, \frac{1}{2}\binom{v-1}{2})$ *designs if and only if* $v \equiv 1$ *or* $2 \pmod 4$*.*

We remark that up to isomorphism, there is exactly one 1-$(5, 3, 3)$ design, given as N_b^+ in Figure 1. It has an automorphism group of order 10, which is transitive on the vertices and on the triples of the design. The uniqueness was verified by an exhaustive computer search. When $v = 6$, the situation is as given by Lemma 4.3, which was also found by an exhaustive computer search.

Lemma 4.3 *Up to isomorphism, there are exactly seven distinct 1-*$(6, 3, 5)$ *designs, given as the columns of the table in Figure 8. Their automorphism groups have the order indicated in the last row of the table.* D_2 *is the complement of* D_1*. The others are self-complementary. There are six ways of partitioning* $\binom{V}{3}$ *into two 1-designs.*

We remark that D_7 is a *twofold triple system*, that is, a 2-$(6, 3, 2)$ design in which each pair of vertices occurs in exactly two triples. Its automorphism group is transitive on its 10 triples, and 2-transitive on its 6 vertices.

Each of $D_1, \ldots, D_6$, together with its complement, contains sub-hypergraphs isomorphic to N_5 and to N_6. D_7 contains an N_5, but no N_6. The following conjecture seems reasonable.

Conjecture 4.4 *Given any 1-*$(v, 3, \frac{1}{2}\binom{v-1}{2})$ *design* H*, where* $v \geq 6$*. The null hypergraph defined by* H *and its complement contains an* N_5 *or an* N_6*.*

D_1	D_2	D_3	D_4	D_5	D_6	D_7
123	123	123	123	123	123	123
124	124	124	124	134	124	124
125	125	125	125	125	125	135
126	134	134	134	134	134	146
134	135	136	136	156	156	156
256	246	245	246	236	236	236
345	256	256	256	246	256	245
346	346	346	345	345	345	256
356	356	356	356	356	346	345
456	456	456	456	456	456	346
8	8	10	2	4	3	60

Figure 8: The seven non-isomorphic 1-$(6, 3, 5)$ designs.

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