A Note on the Georges Configuration and a Paper of Grünbaum

William Kocay*
Department of Computer Science,
Saint Paul's College,
University of Manitoba,
Winnipeg, Manitoba, Canada, R3T 2N2
bkocay@cc.umanitoba.ca

August 25, 2009

Abstract

Grünbaum recently demonstrated the existence of a non-hamiltonian, 3-connected n_3 configuration, with $n = 25$. The configuration is based on the Georges graph. He showed that the configuration can be coordinatized in the real plane, such that all lines can be drawn as straight lines. In this article, we show that the configuration has in fact a coordinatization with rational coordinates. This supports Grünbaum's conjecture that every configuration which has a real coordinatization also has a rational coordinatization.

1 Introduction

In a recent article [2], Grünbaum demonstrated the existence of a non-hamiltonian, 3-connected n_3 configuration, with $n = 25$. The configuration is based on the Georges graph [1]. Grünbaum also showed that the configuration can be coordinatized in the real plane, so that all lines can be drawn as straight lines. The proof was existential, in the sense that it was based on the continuity of the reals

*W. Kocay's research is partially funded by an NSERC discovery grant

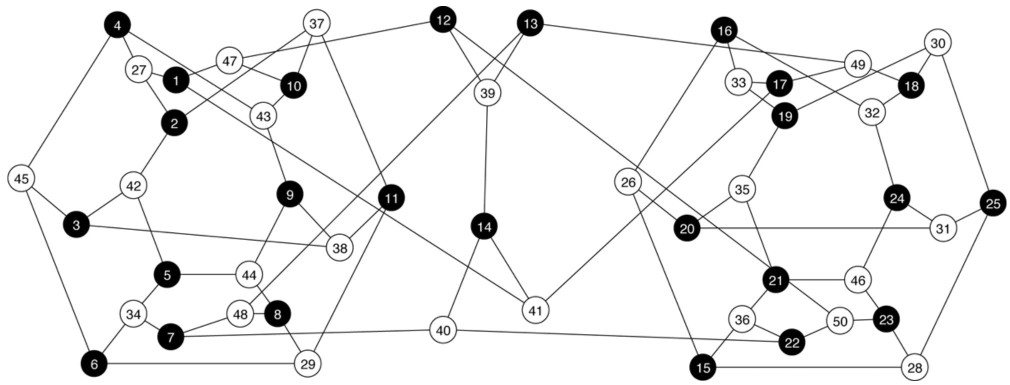

Figure 1: The Georges graph

– it was not shown how to actually construct the coordinatization. In this article, we use determining sets to show that the configuration has in fact a coordinatization with rational coordinates. This supports Grünbaum's conjecture that every configuration which has a real coordinatization also has a rational coordinatization.

The Georges graph is shown in Figure 1. It is a 3-regular, bipartite graph, of girth 6, on 50 vertices. It follows that it is the incidence graph of a 25_3 configuration. It is also non-hamiltonian. The bipartition consists of vertices $1 \dots 25$ (the points) and $26 \dots 50$ (the lines). The vertices representing points in Figure 1 have the same numbering as in [2].

We wish to assign rational coordinates in the real plane to the points $1 \dots 25$ so that the lines $26 \dots 50$ are straight lines. We will use homogeneous coordinates (x, y, z), and construct integer coordinates in the projective plane. Once coordinates have been assigned to collinear points P_i and P_j, this uniquely determines the line containing them – it will have coordinates $P_i \times P_j$; and if coordinates have been assigned to intersecting lines ℓ_i and ℓ_j, this uniquely determines the point of intersection – which will have coordinates $\ell_i \times \ell_j$. Thus, given the coordinates of a collection of independent points and lines, this will determine the coordinates of dependent lines and points, which will in turn determine other points and lines, etc. Following Sturmfels and White [7, 8], we will call this a *construction sequence* for a configuration, since a drawing of the configuration can be constructed by assigning coordinates to the independent points and lines, and then calculating the coordinates of the dependent points and lines, in this order. A minimal set of points and/or lines whose coordinates uniquely determine the coordinates of all points and lines in a configuration is called a *determining set* [3]. It was proved in [3] that if a single incidence is removed from an n_3 configuration, then the resulting configuration always has a determining set. See [3, 4] for more information on determining sets in n_3 configurations.

Let G denote the Georges graph (Figure 1). It is the incidence graph of a 25_3 configuration that we will call the Georges configuration. Remove the incidence between point 2 and line 42 (see Figure 1), and call the resulting configuration G'. A determining set and construction sequence for G' are shown in Figure 2. The vertices representing points are shaded black, the vertices representing lines are shaded white. This determining set was found by the Groups & Graphs software [5] (which can be downloaded from http://www.combinatorialmath.ca). Let P_i denote the coordinates of point i, $(1 \dots 25)$, and let ℓ_i denote the coordinates of line i $(26 \dots 50)$. The determining set consists of vertices $\{2, 4, 6, 9, 11, 23, 26, 30, 33, 39, 41, 48, 28, 46\}$, which appear on the bottom of Figure 2. Arrows are drawn on the edges of the incidence graph to indicate which points and lines determine which others. For example, points 2 and 4 in the determining set determine line 27. Lines 27 and 41 together determine point 1, etc. Notice that line 42 is is the last object in the construction sequence, and that it is determined by points 3 and 5.

Any three non-collinear points in the real projective plane can be mapped to any other three, by a linear transformation. Therefore we are free to take $P_4 = (1, 0, 0), P_{11} = (0, 1, 0)$, and $P_9 = (1, 0, 1)$. Refer to Figure 2. We can also take the $\ell_{41} = (1, 1, 0)$. We want to determine P_2 so that $P_2 \cdot \ell_{42} = 0$. This will ensure that P_2 and ℓ_{42} are collinear. Therefore we set $P_2 = (u, v, w)$, and attempt to determine integer values u, v and w such that $P_2 \cdot \ell_{42} = 0$. The construction sequence implicit in the directed graph of Figure 2 give us the following starting values:

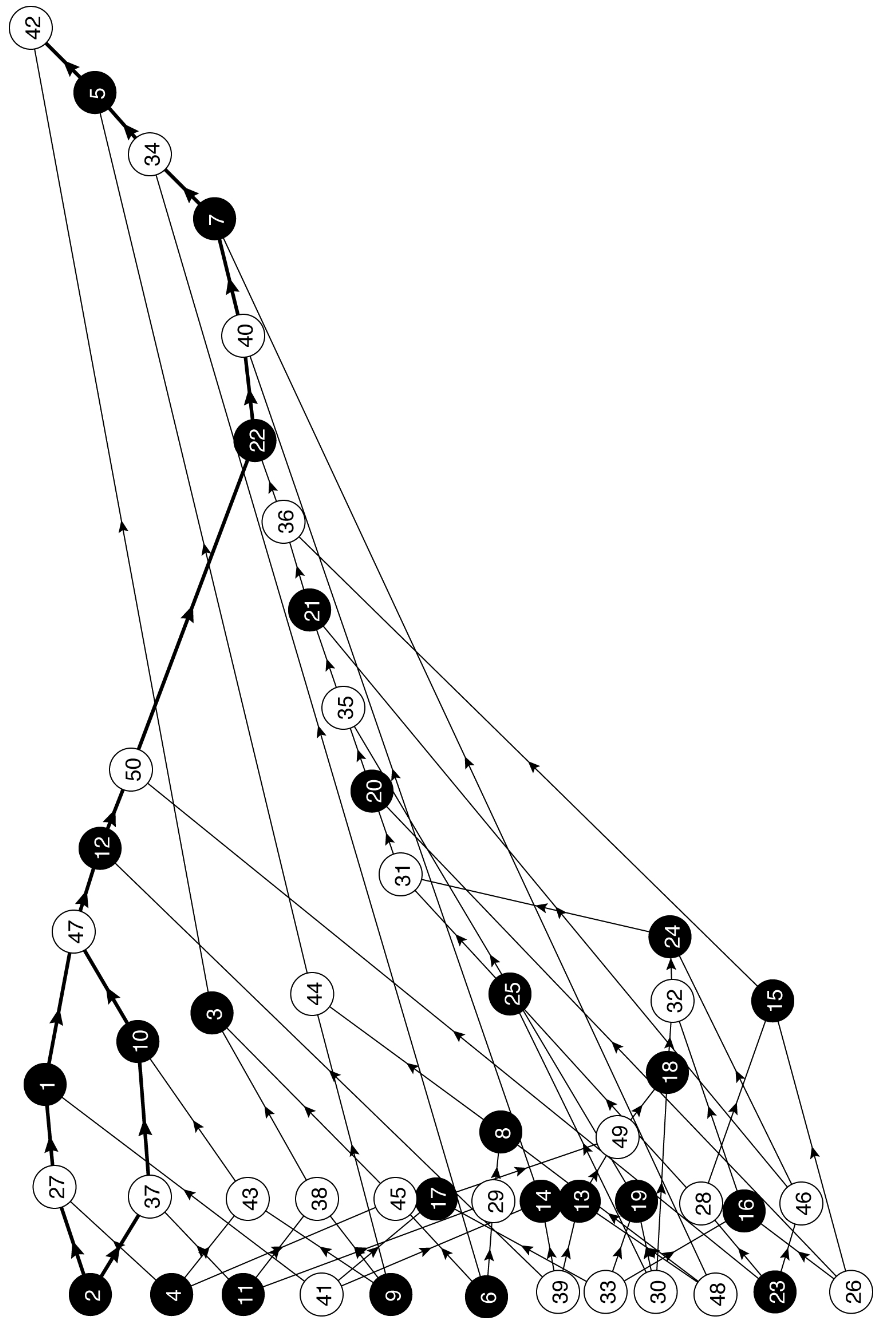

Figure 2: A construction sequence for G'.

point or line	coordinates	definition
P_2	(u, v, w)	$*$
P_4	$(1, 0, 0)$	$*$
P_{11}	$(0, 1, 0)$	$*$
P_9	$(1, 0, 1)$	$*$
ℓ_{41}	$(1, 1, 0)$	$*$
ℓ_{27}	$(0, w, -v)$	$= P_2 \times P_4$
ℓ_{37}	$(-w, 0, u)$	$= P_2 \times P_{11}$
ℓ_{43}	$(0, -1, 0)$	$= P_4 \times P_9$
ℓ_{38}	$(1, 0, -1)$	$= P_{11} \times P_9$
P_1	$(v, -v, -w)$	$= \ell_{27} \times \ell_{41}$
P_{10}	$(u, 0, w)$	$= \ell_{37} \times \ell_{43}$
ℓ_{47}	$(-vw, -w(u+v), uv)$	$= P_1 \times P_{10}$

Table 1: The coordinates of some points and lines

Elements of the determining set are marked with $*$ in the definition column. Their values can be chosen arbitrarily, subject only to the following conditions:

1. unwanted incidences are not allowed, ie, $P_i \cdot \ell_j = 0$ if and only if P_i is incident on ℓ_j;
2. distinct points (lines) must have inequivalent coordinates; P_i and P_j are equivalent if there is a non-zero constant λ such that $P_i = \lambda P_j$.

In the following table, values have been arbitrarily assigned to the remaining points and lines in the determining set. They have been chosen to be as simple as possible, and still satisfy the conditions (1) and (2). The values of the remaining coordinates are then calculated via the construction sequence. When coordinates are calculated by means of a cross product, eg, $P_i = \ell_j \times \ell_k$, if the resulting coordinates of P_i have a common integer factor, then P_i is reduced by that factor. We are also free to multiply P_i by -1, if that is convenient.

Observe that $u, v, w \neq 0$, since if v or w were 0, then $P_2 \cdot \ell_{47} = 0$, which is not possible; if u were 0, then $P_2 \cdot \ell_{29} = 0$, also a contradiction.

Notice that Figure 2 shows that $P_{12}, \ell_{50}, P_{22}, \ell_{40}, P_7, \ell_{34}, P_5$ and ℓ_{42} are the only remaining coordinates that depend on u, v, w.

point or line	coordinates	definition
P_6	$(0,1,1)$	$*$
ℓ_{45}	$(0,-1,1)$	$= P_4 \times P_6$
ℓ_{29}	$(1,0,0)$	$= P_{11} \times P_6$
P_3	$(1,1,1)$	$= \ell_{38} \times \ell_{45}$
ℓ_{33}	$(1,1,1)$	$*$
P_{17}	$(1,-1,0)$	$= \ell_{41} \times \ell_{33}$
ℓ_{39}	$(2,1,1)$	$*$
P_{14}	$(1,-1,-1)$	$= \ell_{41} \times \ell_{39}$
P_{12}	$(-uv - uw - vw, v(2u+w), w(2u+v))$	$= \ell_{47} \times \ell_{39}$
P_{23}	$(1,1,0)$	$*$
ℓ_{50}	$(-w(2u+v), w(2u+v), -3uv - uw - 2vw)$	$= P_{12} \times P_{23}$
ℓ_{30}	$(1,-2,0)$	$*$
P_{19}	$(2,1,-3)$	$= \ell_{33} \times \ell_{30}$
ℓ_{48}	$(1,2,1)$	$*$
P_{13}	$(-1,-1,3)$	$= \ell_{39} \times \ell_{48}$
P_8	$(0,-1,2)$	$= \ell_{29} \times \ell_{48}$
ℓ_{44}	$(1,-2,-1)$	$= P_9 \times P_8$
ℓ_{49}	$(3,3,2)$	$= P_{17} \times P_{13}$
P_{18}	$(4,2,-9)$	$= \ell_{30} \times \ell_{49}$
ℓ_{26}	$(1,2,4)$	$*$
P_{16}	$(2,-3,1)$	$= \ell_{33} \times \ell_{26}$
ℓ_{32}	$(25,22,16)$	$= P_{18} \times P_{16}$

Table 2: The coordinates, continued

Some words about ℓ_{28} and ℓ_{46} are necessary. These lines are part of the determining set, but they are constrained to be incident on P_{23}, ie, $\ell_{28} \cdot P_{23} = 0$ and $\ell_{46} \cdot P_{23} = 0$. Some experimentation was involved in choosing their coordinates, so as to satisfy conditions (1) and (2) above. A number of other choices of their values are also possible.

point or line	coordinates	definition
ℓ_{28}	$(1,-1,4)$	$*$
ℓ_{46}	$(1,-1,3)$	$*$
P_{25}	$(-8,-4,1)$	$=\ell_{30}\times\ell_{28}$
P_{15}	$(-12,0,3)$	$=\ell_{28}\times\ell_{26}$
P_{24}	$(-82,59,47)$	$=\ell_{32}\times\ell_{46}$
ℓ_{31}	$(-247,294,-800)$	$=P_{25}\times P_{24}$
P_{20}	$(694,47,-197)$	$=\ell_{31}\times\ell_{26}$
ℓ_{35}	$(7,211,75)$	$=P_{20}\times P_{19}$
P_{21}	$(354,27,-109)$	$=\ell_{35}\times\ell_{46}$
ℓ_{36}	$(27,82,108)$	$=P_{21}\times P_{15}$
P_{22}	$(246uv+298uw+272vw,$ $-81uv+189uw+54vw,$ $-218uw-109vw)$	$=\ell_{50}\times\ell_{36}$
ℓ_{40}	$(81uv-407uw-163vw,$ $246uv+80uw+163vw,$ $-165uv-487uw-326vw)$	$=P_{22}\times P_{14}$
P_7	$(576uv+1054uw+815vw,$ $-246uv-80uw-163vw,$ $-84uv-894uw-489vw)$	$=\ell_{40}\times\ell_{48}$
ℓ_{34}	$(-162uv+814uw+326vw,$ $-576uv-1054uw-815vw,$ $576uv+1054uw+815vw)$	$=P_7\times P_6$
P_5	$(1728uv+3162uw+2445vw,$ $414uv+1868uw+1141vw,$ $900uv-574uw+163vw)$	$=\ell_{34}\times\ell_{44}$
ℓ_{42}	$(243uv-1221uw-489vw,$ $414uv+1868uw+1141vw,$ $-657uv-647uw-652vw)$	$=P_5\times P_3$

Table 3: The remaining coordinates

The coordinatizing polynomial is given by $P_2\cdot\ell_{42}$, which must equal zero. This is a homogeneous cubic polynomial $P(u,v,w)=$

$$243u^2v-1221u^2w+414uv^2+722uvw-647uw^2+1141v^2w-652vw^2=0$$

In order to find integral solutions to this equation, we use some ideas from elliptic curve theory (see [6]). Experimentation with P_2

shows that if we set $P_2 = (1, -2, -2)$ then an unwanted equivalence $\ell_{47} = \ell_{39} = (2, 1, 1)$ arises, which results in $P_{12} = (0, 0, 0)$, which in turn results in $P_2 \cdot \ell_{42} = 0$. Hence $(u, v, w) = (1, -2, -2)$ is an integral point on the polynomial $P(u, v, w)$. The tangent line to the curve at that point has equation $652u + 407v - 81w = 0$. We solve this for v, then substitute into $P(u, v, w)$ to obtain a cubic in u, w. This cubic is divisible twice by $2u + w$, since $(1, -2, -2)$ is on the curve and on the tangent line. The result is $7734221u - 3886443w = 0$. The corresponding solution is $(u, v, w) = (3886443, -4686705, 7734221)$. If we substitute these numbers into Tables 1, 2 and 3, we obtain the following numeric values for the points and lines whose values were previously given in terms of u, v, w.

point or line	coordinates
P_2	$(3886443, -4686705, 7734221)$
ℓ_{27}	$(0, 7734221, 4686705)$
ℓ_{37}	$(-7734221, 0, 3886443)$
P_1	$(-4686705, 4686705, -7734221)$
P_{10}	$(3886443, 0, 7734221)$
ℓ_{47}	$(1305782643, 1689708754, -677626709)$
P_{12}	$(174859, -520745, 171027)$
ℓ_{50}	$(-57009, 57009, 231868)$
P_{22}	$(-12856204, 12417408, -6213981)$
ℓ_{40}	$(-114303, -116995, 2692)$
P_7	$(-122379, 116995, -111611)$
ℓ_{34}	$(76202, 40793, -40793)$
P_5	$(-40793, 11803, -64399)$
ℓ_{42}	$(-38101, 11803, 26298)$

Table 4: The final numeric coordinates

In order to confirm that conditions (1) and (2) are satisfied by all points and lines, it is best to use a computer. We summarise this with

Theorem 1.1 *The Georges configuration has a coordinatization by rationals.*

References

[1] J.P. Georges, "Non-hamiltonian bicubic graphs", J. Combinatorial Theory B 46 (1989), 121-124.

[2] Branko Grünbaum, "A 3-connected configuration (n_3) with no Hamiltonian circuit", Bulletin of the ICA 46 (2006), 15-26.

[3] W.L. Kocay and R. Szypowski, "The application of determining sets to projective configurations", Ars Combinatoria 53 (1999) 193-207.

[4] W.L. Kocay and Don Tiessen, "Some algorithms for the computer display of geometric constructions in the real projective plane", J. of Comb. Maths. and Comb. Computing, 19 (1995), pp 171-191.

[5] W.L. Kocay, "Groups & Graphs – Software for Graphs, Digraphs, and their Automorphism Groups", Match 58 #2 (2007), 431-443.

[6] Joseph Silverman and John Tate, *Rational Points on Elliptic Curves*, Springer Verlag, 1992, NY.

[7] B. Sturmfels and N. White, "All 11_3- and 12_3-configurations are rational", Aequationes Mathematica 39 (1990), 254-260.

[8] B. Sturmfels and N. White, "Rational realizations of 11_3- and 12_3-configurations", in *Symbolic Computation in Geometry*, H. Crapo et al., IMA preprint series #389, 1988.