On Metamorphosis of Butterfly Factorizations

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Abstract

We have learned that the butterfly factorizations that we described in the January Bulletin of the ICA have appeared in three other forms, and it is useful to give a brief description of these other forms. The earliest paper that we have encountered is a paper on Room squares by Stinson and Wallis that describes houses; their houses are structures equivalent to butterfly factorizations with an additional constraint. Chris Rodger has pointed out that butterfly factorizations are equivalent to the symmetric Latin squares with holes of size two that are discussed in detail in the book by Lindner and Rodger. And Arrigo Bonisoli has sent us a copy of his paper on excessive factorizations that makes an elegant connection with ovals in projective geometries of odd order.

1 Introduction

In the January issue of the Bulletin, we presented a paper [4] on butterfly factorizations, and we later enumerated these factorizations for the case of the complete graph on 8 symbols in [5]. A butterfly factorization of K_8 is shown in Fig. 2. It consists of a central 1-factor (shown in bold), called the *body* 1-factor, and four more pairs of 1-factors, called the left and right *wings*. In general, a butterfly factorization of K_{2n} consists of a 1-factor called the *body* 1-factor and n pairs of 1-factors, the *wings*, corresponding to each body edge, such that each pair of wings has exactly one edge of the body 1-factor in common. Taken together, the edges of the wings cover each edge of K_{2n} – each body edge is covered twice, and all other edges are covered exactly once. Chris Rodger has kindly pointed out to us that the butterfly factorizations that we have introduced are equivalent to the symmetric Latin squares with holes of size two that have been discussed briefly in [2] and in much greater detail in the book by Lindner and Rodger [6]. Arrigo Bonisoli has pointed out the connection with an earlier paper by Stinson and Wallis [7] in which they introduced *houses* as a generalization of Room squares to the case of squares of even side. Arrigo has also drawn our attention to his paper [1] on *excessive factorizations*. So it seems desirable to briefly summarize these different approaches that lead to related results.

2 The Connection with Houses

The paper on Room squares [7] that is closely connected with butterfly factorizations is a very interesting paper by Stinson and Wallis in which they define designs called "houses". Houses are an analogue of Room squares for the case of even sides, and they are defined as follows.

A house is produced in a square of side n, where n is even, by filling the cells of the square either with a blank or with an unordered pair of elements from a set S of n symbols. This selection is similar to that employed for Room squares except that the first two rows of the square must contain the same one-factor F from pairs of elements of S. All other pairs from S are placed in the rows and columns of the square so that every row contains all elements of S, with no repeated pairs, and every column contains all elements of S, with no repeated pairs.

As an example for $n = 6$, we give a diagram of a house. The special one-factor F is $\{12, 34, 56\}$, and it appears in rows one and two. The connection with butterfly factorizations is obvious from the diagram; the first two columns give the left and right wings of a butterfly, the third and fourth columns give the left and right wings of a second butterfly, and the last two columns give the left and right wings of a third butterfly. Note that a house has the additional constraint that each row is also a one-factor of the set S.

12		34		56	
	12		34		56
35		16			24
46			15	23	
	45		26		13
	36	25		14	

Fig. 1, a six-by-six house

Stinson and Wallis give a fairly long construction proving that houses exist for all even values of n greater than 4. Their proof uses the starter-adder construction for some special values of n as well as results on abelian groups. It then uses a recursion involving Room squares and orthogonal Latin squares to get houses for further values of n, up to 116. Finally it gets the entire spectrum for houses by recursion from the values 4,6,8,...,116. The motivation for introducing houses was from a construction for subsquares of Room squares on which Stinson and Wallis were working.

While each house determines a butterfly factorization, it is not known whether the converse is true.

3 The Lindner-Rodger Latin Squares

The Lindner-Rodger symmetric Latin squares with holes of size two look very different from factorizations. So we will give an explanation of the equivalence for the case $n = 8$; this will suffice to indicate the general relationship between butterfly factorizations and the Rodger-Lindner symmetric squares with holes of size two.

Suppose that we start with a butterfly factorization of the complete graph on 8 symbols, say:

37	46	58	**12**	35	47	68
18	26	57	**34**	15	28	67
17	24	38	**56**	13	27	48
16	23	45	**78**	14	25	36

Fig. 2, a butterfly factorization of K_8

Now create a symmetric Latin square of side 8 by the following algorithm. Corresponding to body **12**, make a hole in positions (i, j), where $1 \leq i, j \leq 2$. Place the figure 1 in the cells $(3, 7), (4, 6), (5, 8)$ determined by the left wing of the first butterfly, and their three mirror images $(7, 3), (6, 4), (8, 5)$. Place the figure 2 in the cells $(3, 5), (4, 7), (6, 8)$ determined by the right wing of the first butterfly, as well as their three mirror images $(5, 3), (7, 4), (8, 6)$.

In the next step, corresponding to body **34**, make a hole in positions (i, j), where $3 \leq i, j \leq 4$. Place the figure 3 in cells $(1, 8), (2, 6), (5, 7)$ and in their three mirror images; place the figure 4 in cells $(1, 5), (2, 8), (6, 7)$ and in their three mirror images.

Follow the same procedure for the bodies **56** and **78**. The result is the following symmetric Latin square with four holes down the principal diagonal. (We note that the holes could be filled in with 2×2 latin subsquares, if desired.)

*	*	6	8	4	7	5	3
*	*	7	5	8	3	6	4
6	7	*	*	2	8	1	5
8	5	*	*	7	1	2	6
4	8	2	7	*	*	3	1
7	3	8	1	*	*	4	2
5	6	1	2	3	4	*	*
3	4	5	6	1	2	*	*

Fig. 3, a symmetric Latin square with holes of size 2

The reverse procedure is obvious. If we start from a symmetric Latin square with holes of size two, such as the one just created, then we can immediately write down the $n/2$ bodies by using the positions of the holes. For the body ab, we get the left wing by taking the names of the cells that contain symbol a; since each cell appears twice in the forms (x, y) and (y, x), we need merely take the pair (x, y). We get the right wing by collecting the names of the cells that contain the symbol b.

Thus we have an exact correspondence between butterfly factorizations and symmetric Latin squares with holes of size two. To speak biologically, the butterflies have metamorphosed into a symmetric Latin square with holes of size two. We note that there is a similar correspondence between 1-factorizations of K_{2n} and symmetric latin squares with $2n$ on the diagonal.

4 The Excessive Factorizations of Bonisoli

Bonisoli defines an *excessive factorization* of K_{2n} as a collection of $2n$ 1-factors such that every edge of K_{2n} is covered either once or twice, and furthermore, the edges covered twice form a 1-factor. Clearly this definition includes all butterfly factorizations. However an excessive factorization need not have a partition into wings. It is possible that some 1-factors contain two or more edges of the body 1-factor, and that some 1-factors contain no edges of the body 1-factor. Bonisoli proves the existence of cyclic excessive factorizations of K_{2n} for all $n > 2$. The proof is based on techniques similar to those used by Hartman and Rosa [3] to construct cyclic 1-factorizations of K_{2n}. Some of the factorizations used are not butterfly factorizations. Bonisoli's paper is also noteworthy for the interesting connection that it makes with ovals in projective planes of odd order.

5 Houses and K_8 Factorizations

There are two non-isomorphic butterfly factorizations of K_8, as shown in [5]. A house for one of them appears in [7]. In fact, both factorizations have houses, as shown by the following diagrams.

23	46	78	**15**	28	34	67
18	35	47	**26**	17	38	45
14	25	68	**37**	16	24	58
12	36	57	**48**	13	27	56

15		26		37		48	
	15		26		37		48
23	67	18	45				
46	28	35	17				
78	34			25	16		
		47	38			12	56
				14	58	36	27
				68	24	57	13

Fig. 4, The factorization of K_8 with automorphism group of order 48

24	38	67	**15**	23	46	78
18	35	47	**26**	17	34	58
16	28	45	**37**	14	25	68
13	27	56	**48**	12	36	57

15		26		37		48	
	15		26		37		48
67	23	18		45			
			58		14	27	36
	46			28		13	57
24		35	17		68		
38		47				56	12
	78		34	16	25		

Fig. 5, The factorization of K_8 with automorphism group of order 12

It is not known whether every butterfly factorization is contained in a house.

References

[1] A. Bonisoli, Edge Covers of K_{2n} with $2n$ One-factors, *Rendiconti del Seminario Matematico di Messina, Series II*, Tomo XXV, Volumen 9 (2003).

[2] C.M. Fu, H.L. Fu, S. Milici, G. Quattrochi, and C.A. Rodger, Almost resolvable directed $2r$-cycle systems, *J. of Combinatorial Design* **3** (1995), no. 6, 443-447.

[3] A. Hartman, A. Rosa, Cyclic One-Factorization of the Complete Graph, *Europ. J. Combinatorics* **6** (1985), 4548.

[4] W.L. Kocay and R.G. Stanton, On Butterfly Factorizations, *Bulletin of the Institute of Combinatorics and its Applications* **55** (2009), 57–65.

[5] W.L. Kocay and R.G. Stanton, Butterfly Factorizations of K_8, *Utilitas Mathematica* **79** (2009), 221-225.

[6] C.C. Lindner and C.A. Rodger, *Design Theory*, CRC Press, 1997.

[7] D.R. Stinson and W.D. Wallis, An even-side analogue of Room squares, *Aequationes Mathematicae* **27** (1984), 201-213.