A 10_3 Configuration that is almost a Theorem

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Abstract

There are ten 10_3 configurations, of which only one, the Desargues configuration is a theorem. It is shown that another 10_3 configuration is *almost* a theorem.

1 The 10_3 Configurations

An n_3 configuration is an incidence structure consisting of n points and n lines, such that each point is incident on three lines, and each line is incident on three points. Furthermore, any two distinct lines intersect in at most one point. It is well known that there are ten distinct 10_3 configurations [1, 2, 3, 5]. A configuration is said to be *drawn in the plane*, if its lines are represented as distinct straight lines in the plane, which intersect appropriately at the points of the configurations.

Definition 1 *An n_3 configuration is said to be a* theorem*, if when any single incidence is removed, the resulting configuration can be drawn in the plane, and the missing incidence is always satisfied.*

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For example, the Pappus configuration is a 9_3 configuration that is a theorem. Pappus's theorem guarantees that the missing incidence is always satisfied. The Desargues configuration is a 10_3 configuration that is also a theorem. The Fano configuration is a 7_3 configuration that is not almost a theorem – if one incidence is removed it can be drawn in the real plane, but the missing incidence is never satisfied. Another 10_3 configuration, the *anti-Pappian*, [2] cannot be drawn in any plane with straight lines [4, 5]. It is not almost a theorem, although it can be drawn in the plane if any incidence is removed.

The remaining eight 10_3 configurations can all be drawn in the plane, but unlike the Desargues configuration, the coordinates of the points must be carefully chosen so that all the incidences are satisfied — they are not theorems. Bokowski and Sturmfels [1] have found rational coordinatizations of all 10_3 configurations that can be drawn in the plane. Following Schröter [5], Grünbaum [3] describes constructions for the 10_3 configurations.

In this article we show that one of the 10_3 configurations is *almost a theorem* — if it is drawn in the plane so that all incidences but one are satisfied, then under a very simple condition the missing incidence is also satisfied.

The incidence structure of the 10_3 configuration is given in the following table. It is isomorphic to configuration number $(10_3)_2$ in the book [3] (p. 73) by Grünbaum. The points are here labelled $A, B, C, D, E, P, Q, R, S, T$ (corresponding to points $9, 8, 3, 1, 0, 4, 5, 2, 7, 6$, respectively, of [3]), and the lines are labelled $1, 2, \ldots, 10$.

line	1	2	3	4	5	6	7	8	9	10
	D	D	D	B	R	C	R	C	P	Q
	R	P	T	A	P	S	T	Q	T	S
	C	Q	S	E	B	B	A	A	E	E

Let A, B, C, D be any four points in the projective plane, no three collinear (see Figure 1). Construct the lines AB, AC, BC, CD. Choose E to be the *harmonic conjugate* of the intersection $AB \cap CD$ with respect to points A and B. Now choose any point P in the plane *arbitrarily*, so long as P is not on the lines $AB, AC, BC, CD, BD, DE,$ AD. This restriction is only to ensure that the construction produces 10 distinct points and 10 distinct lines.

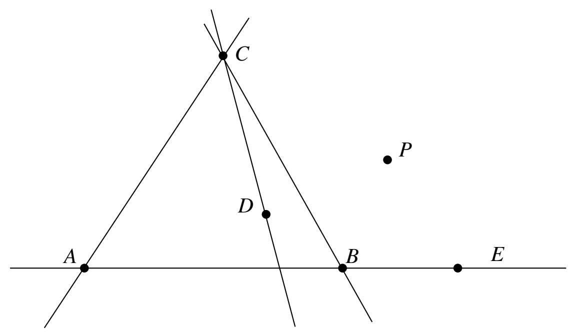

Figure 1: Choose A, B, C, D

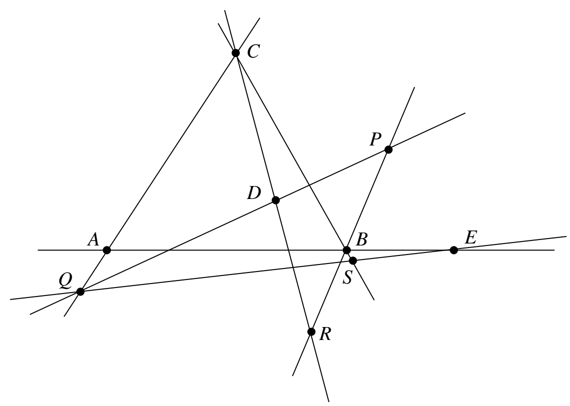

Figure 2: Construct Q, R, S

Now construct the following sequence of lines and points:

line PD, point $Q = AC \cap PD$
line PB, point $R = CD \cap PB$
line QE, point $S = CB \cap QE$

This is illustrated in Figure 2.

Now construct the lines AR and DS and let $T = AR \cap DS$. Construct the line ET. This is illustrated in Figure 3.

Theorem 1 *Suppose that E is chosen as the harmonic conjugate of $AB \cap CD$ on line AB. Then point P is incident with line ET. The resulting configuration is a planar drawing of a 10_3 configuration.*

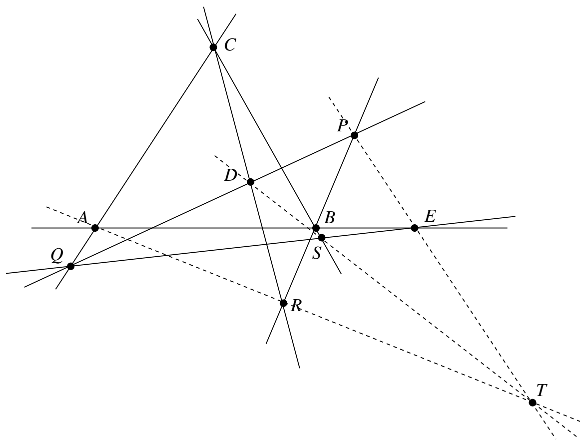

Figure 3: Construct ET

Proof. The proof is analytic, using homogeneous coordinates in the projective plane. In the real projective plane, any four points, no three collinear, can be mapped to any other four points, no three collinear. Therefore we are free to choose the following coordinates for A, B, C, D.

$A = (1, 0, 0)$
$B = (0, 1, 0)$
$C = (0, 0, 1)$
$D = (1, 1, 1)$.

The lines AB and CD intersect at point $(1, 1, 0)$, which has harmonic conjugate $E = (1, -1, 0)$ with respect to A and B. As P is arbitrary, let its coordinates be $P = (u, v, w)$. We then calculate the coordinates of

$Q = AC \cap PD = (u - v, 0, w - v)$
$R = CD \cap PB = (u, u, w)$
$S = CB \cap QE = (0, v - u, v - w)$

Now $u \neq v$, for otherwise $Q = C$, which can only occur if P is on the line CD. However P was chosen not on CD. Also $v \neq w$, for otherwise $Q = A$, which can only occur if P is on the line AD. However P was chosen not on AD.

Calculating the lines AR and DS and their intersection gives

$$T = (v(u-w), u(u-w), w(u-w)) = (u-w)(v, u, w)$$

As before, $u - w \neq 0$; for if $u = w$, then $R = u(1,1,1) = uD$, so that R and D are the same point, which means that P is on the line BD, contradicting the choice of P. Hence, the coordinates of T can be taken as (v, u, w). The determinant of the coordinates of E, P and T is then

$$\begin{vmatrix} 1 & -1 & 0 \\ u & v & w \\ v & u & w \end{vmatrix} = 0$$

proving that E, T and P are always collinear. It is straightforward to verify that the 10 points and 10 lines are distinct, so that the construction is a planar drawing of a 10_3 configuration. It is configuration $(10_3)_2$ of [3].

It is clear that if u, v, w are chosen to be integers, then all the points have integer coordinates. Thus although it is can be fairly difficult to find a rational coordinatization of an arbitrary 10_3 configuration that is not a theorem, by choosing E as the harmonic conjugate of $AB \cap CD$, the missing incidence is *always satisfied*, so that it is very easy to construct rational coordinatizations of this configuration, since the harmonic conjugate of rational coordinates is again rational.

2 A Synthetic Proof

Harmonic conjugates are defined synthetically in terms of a quadrilateral. Let L_1, L_2, L_3, L_4 be any four distinct lines, no three concurrent, in the projective plane. Let $X = L_1 \cap L_2$ and $Y = L_3 \cap L_4$ be any two of the intersection points. There are four remaining points of intersection of the four lines, which determine two diagonals, shown as dashed lines in Figure 5. The two diagonals intersect the line XY

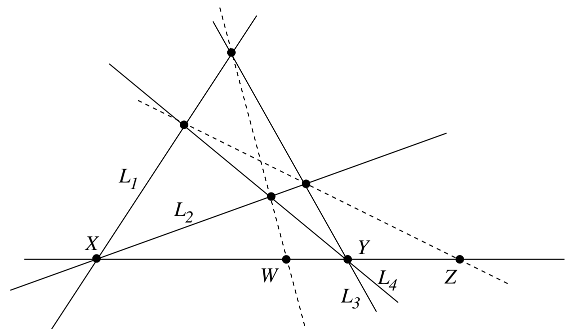

Figure 4: Harmonic Conjugates $(X, Y; W, Z)$

in points W and Z which are harmonic conjugates with respect to X and Y. The relation of being harmonic conjugates is a projective invariant.

Theorem 2 *In the construction of Figure 3, line PT contains point E.*

Proof. Let V denote the intersection $AB \cap CD$ in Figure 3. Let $X = AR \cap QD$ and $Y = BR \cap SD$. Consider the triangles ΔQAX and ΔSBY. The sides QA and SB intersect in C; the sides AX and BY intersect in R; the sides QX and SY intersect in D. Now C, D and R are collinear, so that by Desargues's theorem, the triangles are in perspective. Hence lines QS, AB and XY are concurrent. Their intersection point must be E. Therefore X, Y and E are collinear. Projecting the line AB from R onto line XY (which is not shown in the diagram), we see that points E and $CR \cap XY$ are harmonic conjugates with respect to X and Y.

Now consider the quadrilateral consisting of lines AX, QX, and RY, DY. The line XY contains two of the intersection points of the quadrilateral. The two diagonals are the lines DR and PT, which intersect XY in points that are harmonic conjugates with respect to X and Y. It follows that line PT contains E.

It follows from this proof that in *any* drawing of this configuration in the plane, with the points here labelled $A, B, C, D, E, P, Q, R, S, T$, that E and $CR \cap AB$ must be harmonic conjugates with respect to A and B.

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